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Untangling circular drawings: Algorithms and complexity

Sujoy Bhore [a], Guangping Li [b,*], Martin Nöllenburg [b], Ignaz Rutter [c], Hsiang-Yun Wu [d,e]

[a] *Indian Institute of Technology Bombay, Mumbai, India*
[b] *Algorithms and Complexity Group, TU Wien, Vienna, Austria*
[c] *University of Passau, Passau, Germany*
[d] *St. Pölten University of Applied Sciences, St. Pölten, Austria*
[e] *Research Unit of Computer Graphics, TU Wien, Vienna, Austria*

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ABSTRACT

We consider the problem of untangling a given (non-planar) straight-line circular drawing δ_G of an outerplanar graph $G = (V, E)$ into a planar straight-line circular drawing of G by shifting a minimum number of vertices to a new position on the circle. For an outerplanar graph G, it is obvious that such a crossing-free circular drawing always exists and we define the *circular shifting number* $\text{shift}^{\circ}(\delta_G)$ as the minimum number of vertices that are required to be shifted in order to resolve all crossings of δ_G. We show that the problem Circular Untangling, asking whether $\text{shift}^{\circ}(\delta_G) \le K$ for a given integer K, is NP-complete. For n-vertex outerplanar graphs, we obtain a tight upper bound of $\text{shift}^{\circ}(\delta_G) \le n - \lfloor\sqrt{n-2}\rfloor - 2$. Moreover, we study the Circular Untangling for *almost-planar* circular drawings, in which a single edge is involved in all of the crossings. For this problem, we provide a tight upper bound $\text{shift}^{\circ}(\delta_G) \le \lfloor \frac{n}{2} \rfloor - 1$ and present an $O(n^2)$-time algorithm to compute the circular shifting number of almost-planar drawings.

1. Introduction

The family of outerplanar graphs, i.e., the graphs that admit a planar drawing where all vertices are incident to the outer face, is an important subclass of planar graphs and exhibits interesting properties in algorithm design, e.g., they have treewidth at most 2. Being defined by the existence of a certain type of drawing, the study of outerplanar graphs is a fundamental topic in the field of graph drawing and information visualization; they are relevant to circular graph drawing [28] and book embedding [3,5]. Several aspects of outerplanar graphs have been studied over the years, e.g., characterization [8,13,29], recognition [1,31], and drawing [14,21,27]. Moreover, outerplanar graphs and their drawings have been applied to various scientific fields, e.g., network routing [15], VLSI design [9], and biological data modeling and visualization [20,32].

In this paper, we study the untangling problem for non-planar circular drawings of outerplanar graphs, i.e., we look for the minimum number of vertices needed to shift in order to turn the given non-planar circular straight-line drawing into a planar one. Similar untangling ideas have been used previously to eliminate edge crossings in non-planar drawings of planar graphs [17]. More precisely, let $G = (V, E)$ be an n-vertex outerplanar graph and let δ_G be an outerplanar drawing of G, which can be described combinatorially as the (cyclic) order $\sigma = (v_1, v_2, \dots, v_n)$ of V when traversing vertices on the

* Corresponding author.
E-mail address: guangping.li@tu-dortmund.de (G. Li).

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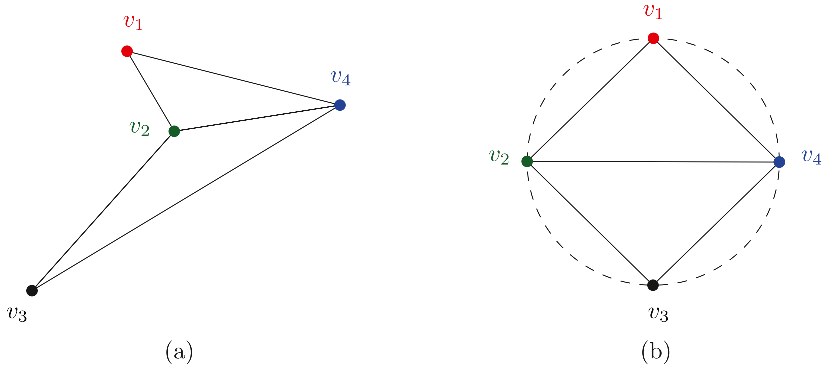


Fig. 1. Morphing an outerplanar drawing (a) into a circular drawing (b).

boundary of the outer face counterclockwise. This order σ corresponds to a planar circular drawing by mapping each vertex $v_i \in V$ to the point p_i on the unit circle $\mathcal{O}$ with the polar coordinate $p_i = (1, \frac{i}{n} \cdot 2\pi)$ and drawing each edge $(v_i, v_j) \in E$ as the straight-line segment between its endpoints p_i and p_j; see Fig. 1.

We note that it is sufficient to consider circular drawings since any outerplanar drawing can be transformed into an equivalent circular drawing by morphing the boundary of the outer face to $\mathcal{O}$ and then redrawing the edges as straight segments [28]. Assume the target drawing is known and the corresponding vertices are labeled. The transformation of the current circular vertex ordering to the target circular vertex ordering with minimal vertex moves, is identical to finding the longest common subsequence between two cyclic permutations [24].

Our untangling problem is further motivated by the problem of maintaining an outerplanar drawing of a *dynamic* outerplanar graph, which is subject to edge or vertex insertions and deletions, while maximizing the visual *stability* of the drawing [22,23], i.e., the number of vertices that can remain in their current position. Such problems of maintaining drawings with specific properties for dynamic graphs have been studied before [2,4,11,12], but not for the outerplanarity property.

Related work. The notion of untangling is often used in the literature for a crossing elimination procedure that makes a non-planar drawing of a planar graph crossing-free; see [10,18,25,26]. Given a straight-line drawing δ_G of a planar graph G, the problem to decide whether it is possible to untangle δ_G by moving at most K vertices, is known to be NP-hard [17,30]. Lower bounds on the number of vertices that can remain fixed in an untangling process have also been studied [6,7,17]. On the one hand, Bose et al. [6] proved that $\Omega(n^{1/4})$ vertices can remain fixed when untangling a drawing. On the other hand, Cano et al. [7] gave a family of drawings, where at most $O(n^{0.4948})$ vertices can remain fixed during an untangling process. Goaoc et al. [17] proposed an algorithm, which allows at least $\sqrt{(\log n - 1)/\log\log n}$ vertices to remain fixed when untangling a drawing. Given an arbitrary drawing of an n-vertex outerplanar graph, all edge crossings can be eliminated while keeping at least $\sqrt{n/2}$ vertices fixed [17,26], whereas there exists a drawing δ_G of an n-vertex outerplanar graph G such that at most $\sqrt{n-1}+1$ vertices can stay fixed when untangling δ_G [17]. Kraaijer et al. [19] proposed several variants on untangling moves such as swapping the locations of two adjacent vertices or rotating an edge over 90 degrees. They showed that it is NP-complete to decide if a drawing can be untangled by swapping. They also proved that to minimize the number of swaps needed to untangle an embedded tree is NP-hard.

Note that the untangled drawings in these previous works are planar but not necessary outerplanar. In this paper, we study untanglings transforming non-outerplanar circular drawings into outerplanar circular drawings.

Preliminaries and problem definition. Given a graph $G = (V, E)$, a 2-connected component of G is a maximal subgraph of G such that after removing any single vertex of G, the subgraph remains connected. Two subsets $A, B \subseteq V$ are *adjacent* if there is an edge $ab \in E$ with $a \in A$ and $b \in B$. A *bridge* (resp. *cut-vertex*) of G is an edge (resp. vertex) whose deletion increases the number of connected components of G.

A drawing of a graph is *planar* if it has no crossings, it is *almost-planar* if there is a single edge that is involved in all crossings, and it is *outerplanar* if it is planar and all vertices are incident to the outer face. A graph $G = (V, E)$ is *outerplanar* if it admits an outerplanar drawing and it is known that such a drawing exists if and only if G has neither K_4 nor $K_{2,3}$ as a minor. A drawing where the vertices lie on a circle and the edges are drawn as straight-line segments is called a *circular drawing*. Every outerplanar graph G admits a planar circular drawing, as one can start with an arbitrary outerplanar drawing δ_G of G and transform the boundary of the outer face of δ_G to a circle [28]. In this paper, we exclusively work with circular drawings of outerplanar graphs; we thus refer to them as drawings. Let $e = uv$ be an edge of a graph G and let δ_G be a circular drawing of G. For simplicity, we refer to the line segment uv in the drawing δ_G as the edge e of δ_G.

Given a non-planar circular drawing δ_G of an n-vertex outerplanar graph G where the vertices lie on the unit circle $\mathcal{O}$, we can transform the drawing δ_G to a planar circular drawing by moving the vertices on the circle $\mathcal{O}$. Formally, given a circular drawing δ_G, a vertex move operation (or shift) changes the position of one vertex in δ_G to another position on the circle $\mathcal{O}$ [17]. We call a sequence of moving operations that results in a planar circular drawing an *untangling* of δ_G. We say an untangling is *minimum* if the number of vertex moves of this untangling is the minimum over all untanglings of δ_G. We define the *circular shifting number* $\mathrm{shift}^{\circ}(\delta_G)$ of a circular drawing δ_G as the number of vertex moves in a minimum untangling of δ_G. Now we can formulate the relevant problems.

Problem 1.1 *(*CIRCULAR UNTANGLING (CU)*).* Given a circular drawing δ_G of an outerplanar graph G and an integer K, decide if $\mathrm{shift}^{\circ}(\delta_G) \le K$.

Problem 1.2 *(*MINIMUM CIRCULAR UNTANGLING (MINCU)*).* Given a circular drawing δ_G of an outerplanar graph G, find an untangling of δ_G with $\mathrm{shift}^{\circ}(\delta_G)$ vertex moves.

Contributions. We show that CIRCULAR UNTANGLING is NP-complete in Section 2. Then, in Section 3, we provide a tight upper bound of the circular shifting number. Next, we consider almost-planar drawings. In this case, we provide a tight upper bound on the circular shifting number in Section 4 and design a quadratic-time algorithm to compute a minimum untangling in Section 5.

2. Complexity of CIRCULAR UNTANGLING

In this section, we prove the following theorem.

Theorem 2.1. CIRCULAR UNTANGLING *is* NP-*complete.*

The NP-hardness follows by a reduction from the well-known NP-complete problem 3-PARTITION [16]. However, we do not give a direct reduction but rather work via an intermediate problem, called DISTINCT INCREASING CHUNK ORDERING WITH REVERSALS that concerns increasing subsequences. A *chunk* C is a sequence $C = (c_i)_{i=1}^{n}$ of positive integers. For a chunk C, we define $C^1 = C$, and we denote its reversal by C^{-1}. We introduce the following problem.

Problem 2.2 *(*INCREASING CHUNK ORDERING WITH REVERSALS (ICOR)*).* Given a multiset $\mathcal{C} = \{C_1, \dots, C_\ell\}$ of ℓ *chunks* and a positive integer M, determine whether a permutation π of $\{1, \dots, \ell\}$ and a function $\varepsilon \colon \{1, \dots, \ell\} \to \{-1, 1\}$ exist such that the concatenation $C_{\pi(1)}^{\varepsilon(1)} C_{\pi(2)}^{\varepsilon(2)}, \dots, C_{\pi(\ell)}^{\varepsilon(n)}$ contains a strictly increasing subsequence of length M.

This problem also comes in a *distinct* variant, denoted DIST-ICOR, where all integers in all input chunks are required to be distinct. We first show that DIST-ICOR is NP-complete and then reduce it to CIRCULAR UNTANGLING. Since we feel that DIST-ICOR may serve as a useful starting point for future reductions, we explicitly state our intermediate result.

Theorem 2.3. DIST-ICOR *is* NP-*complete.*

2.1. Proof of Theorem 2.3

Observe that DIST-ICOR lies in NP, since we can non-deterministically guess an ordering of chunks and whether each of them is reversed or not. We can further guess an increasing subsequence of the concatenation and check its length. The remainder of this section is devoted to showing NP-hardness by giving a reduction from 3-PARTITION.

The input to the 3-PARTITION problem consists of a multiset $A = \{a_1, \dots, a_{3m}\}$ of $3m$ positive integers and a positive integer K such that $\frac{K}{4} < a_i < \frac{K}{2}$ for $i = 1, \dots, 3m$. The question is whether A can be partitioned into m disjoint triplets $T_1, \dots, T_m$ such that $\sum_{a \in T_j} a = K$ for all $j = 1, \dots, m$. It is well-known that 3-PARTITION is strongly NP-complete, i.e., the problem is NP-complete even if the integers in A and K are polynomially bounded in m [16].

Let $I = (A, K)$ with $A = \{a_1, \dots, a_{3m}\}$ be an instance of 3-PARTITION. We assume that each number in A is a multiple of $3m$, otherwise, we can multiply each element in A and K by $3m$. We now construct an instance $I' = (\mathcal{C}, M)$ of DIST-ICOR in polynomial time.

Construction. We create for each element $a_i \in A$ a corresponding chunk C_i as follows. For two integers a, l, we denote the consecutive integer sequence $(a, a+1, \dots, a+l-1)$ as the *incremental sequence* of length l starting at a. We say that a sequence of integers *crosses* an integer c if it contains both a number that is at most c and a number that is at least $c+1$. Let $X = 3mK$. We take all incremental sequences of length $a_i + X$ starting at $\alpha \cdot (K + 3X) + \beta \cdot X + \gamma$ for $\alpha \in \{0, \cdots m-1\}$, $\beta \in \{0, 1, 2\}$ and $\gamma \in \{1, 2, \cdots, K - a_i\}$. Note that there are at most X such sequences and no such sequence crosses a multiple of $K + 3X$. To construct the chunk C_i, we first build a chunk C'_i with possibly repeating numbers as follows. The chunk C'_i is formed by concatenating all these incremental sequences of length $a_i + X$ in decreasing order of their starting number; see Fig. 2. Observe that, in the figure, a strictly increasing subsequence corresponds to a path of segments with positive slopes, whereas a non-increasing subsequence corresponds to a path with non-positive slopes.

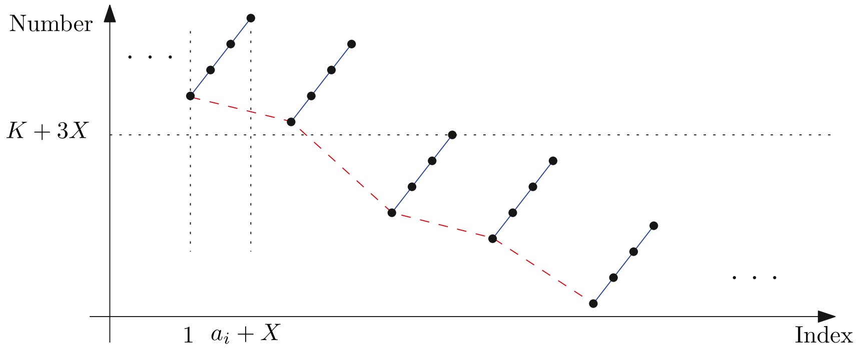


Fig. 2. Construction of chunk C_i as a concatenation of incremental sequences of length a_i+X in the decreasing order of their first number. Each blue solid path corresponds to an incremental sequence. The red dashed path, which connects the first numbers of the incremental sequences, slopes downward. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

To make the elements distinct, we introduce pairs of numbers, which we order lexicographically. We take the concatenation C of chunks $C'_1, C'_2, \cdots, C'_{3m}$, then replace the number a at the i-th position by the pair $(a, |C|-i)$ for each position i, where $|C|$ is the length of the sequence C. The chunks $C_1, \ldots, C_{3m}$ of number pairs are obtained by cutting this modified sequence C of number pairs in such a way that $|C_i| = |C'_i|$ for $i = 1, \ldots, 3m$. To get an instance with chunks of numbers again, at the end of the construction, each number pair is replaced by its rank in a lexicographically increasing ordering of all pairs that occur in the instance. For simplicity, we use the construction with pairs in the following. We obtain an instance $I' = (\mathcal{C}, M)$ of Increasing Chunk Ordering with Reversals by setting $\mathcal{C} = \{C_1, \ldots, C_{3m}\}$ and $M := m(K+3X)$.

For a sequence of pairs with two entries, we call the sequence obtained by keeping only the first entry of each pair, its *projection*. Note that the projection of C_i is C'_i.

Lemma 2.4. *For each $i \in \{1, \ldots, 3m\}$, the chunk C_i has the following properties:*

(i) For every strictly increasing subsequence of C_i, its projection is a strictly increasing sequence with respect to the lexicographic ordering of pairs.
(ii) No projection of a strictly increasing subsequence of C_i crosses a multiple of $K+3X$.
(iii) For each $\alpha \in \{0, \cdots, m-1\}$, $\beta \in \{0, 1, 2\}$ and $\gamma \in \{1, 2, \cdots, K-a_i\}$, there exists a subsequence of C_i whose projection is the incremental sequence of length $a_i + X$ starting at $\alpha \cdot (K+3X) + \beta \cdot X + \gamma$.
(iv) Every strictly increasing subsequence of C_i has length at most $a_i + X$.
(v) Every strictly increasing subsequence of C_i^{-1} has length at most X.

Proof. Since the sequence obtained by keeping the second entry of each pair of C_i is strictly decreasing, we get Property (i). Property (ii) and Property (iii) follow directly from the construction of C'_i.

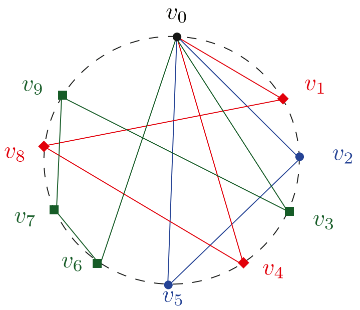
To show Property (iv), suppose for a contradiction that there exists a strictly increasing subsequence s of C_i such that the length of s is bigger than $a_i + X$. By Property (i), the projection s' of s is a strictly increasing sequence of C_i. Since C'_i is a concatenation of incremental sequences of length $a_i + X$ (in the decreasing order of their starting number), there exists an index $j \in \{1, ..., a_i + X\}$ such that s' contains the j-th elements of two incremental subsequences of C'_i. By the construction of C'_i, these elements are in the decreasing order in C'_i, a contradiction.

For Property (v), consider a strictly increasing subsequence of C_i^{-1}. It corresponds to a strictly decreasing subsequence s of C_i, and its projection s' is a non-increasing subsequence of C'_i. Note that s' contains at most one element of each incremental sequence of C'_i, and C'_i is the concatenation of at most X incremental sequences. Therefore, the length of s' is at most X. $\square$

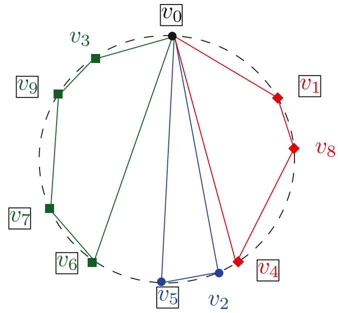
In order to finish the proof of Theorem 2.3 it remains to show the following.

Lemma 2.5. *I' is a yes-instance of* Dist-ICOR *if and only if I is a yes-instance of* 3-Partition.

Proof. Assume there is a partition of the elements of A into m triplets, each of which sums to K. We arbitrarily order these triples, and within each triplet, we order the elements according to their index. This defines a total ordering on the elements, and therefore on the chunks. Let $T_i = \{a_x, a_y, a_z\}$ with $x < y < z$ be the ith triplet and let C_x, C_y, C_z be the corresponding chunks. By Property (iii), C_x, C_y, and C_z contain, respectively, three subsequences whose projections are the incremental sequences of length $a_x + X$, $a_y + X$, and $a_z + X$ starting at $(i-1)(K+3X)+1$, $(i-1)(K+3X)+X+a_x+1$, and $(i-1)(K+3X)+2X+a_x+a_y+1$. Concatenating these subsequences for all chunks hence gives an increasing subsequence whose projection is the sequence $1, \cdots, m(K+3X)$.



(a) The straight-line circular drawing δ_G constructed from a DIST-ICOR instance with chunks $C_1 = (2, 5)$, $C_2 = (1, 8, 4)$, and $C_3 = (6, 7, 9, 3)$.



(b) An example drawing obtained by applying an optimum untangling on δ_G. Fixed vertices are marked in $\square$.

Fig. 3. The reduction from DIST-ICOR to CIRCULAR UNTANGLING.

Conversely, assume that there is a chunk ordering that contains a strictly increasing subsequence S of length $m(K + 3X)$. By Property (iv) and Property (v), each chunk C_i or its reversal can contribute a subsequence of at most $a_i + X$ elements, therefore each chunk C_i or its reversal must contribute an increasing subsequence of length $a_i + X$. Moreover, reversing C_i only provides a shorter increasing subsequence than $a_i + X$, thus no C_i is reversed. We cut the sequence S into m consecutive sequences $S_1, S_2 \dots S_m$, called *partition cells* of S, such that the projection of S_i consists of numbers in $\{(i-1)(K+3X)+1, \cdots, i(K+3X)\}$. By Property (ii), the projection of every strictly increasing subsequence inside a chunk does not cross a multiple of $K + 3X$, thus each chunk contributes to exactly one partition cell. We claim the following:

Claim 1. Each partition cell has length $K + 3X$.

We first show how the proof of the lemma can be derived from the claim. Since the length of each cell is $K + 3X$, exactly three chunks contribute to each cell. Each such triplet of chunks then corresponds to a triplet of A whose sum is K. Together, these triplets provide a solution of the instance I of 3-PARTITION.

It remains to prove the claim. Consider a partition cell S_i consisting of numbers from n chunks. Then S_i is the concatenation of subsequences $S_{i,1}, S_{i,2}, \dots, S_{i,n}$, $n \le 3m$, each of which is contributed by a different chunk. Since the projection of S_i is a non-decreasing sequence consisting of numbers in $\{(i-1)(K+3X)+1, \cdots, i(K+3X)\}$ and by Property (i), the projection of each $S_{i,j}$ is a strictly increasing sequence, it follows that non-strict increases of S_i can only occur when moving from $S_{i,j}$ to $S_{i,j+1}$ for some j. Thus, $|S_i| < K + 3X + n \le K + 3X + 3m$.

Note that X, K and $|S_i|$ are all multiples of $3m$. For X, this is by definition, for K, it follows from the fact that each element of A is a multiple of $3m$, and for $|S_i|$ recall that each chunk C_j that contributes a nonempty subsequence of S_i contributes a sequence of length $X + a_j$. Therefore $|S_i| < K + 3X + 3m$ implies $|S_i| \le K + 3X$. Suppose there exists a partition cell S_j with $|S_j| < K + 3X$, then $|S| < m(K + 3X)$, which contradicts our assumption of $|S| = m(K + 3X)$. Hence $|S_i| = K + 3X$ as claimed. $\square$

2.2. Proof of Theorem 2.1

It is readily seen that CIRCULAR UNTANGLING lies in NP. So it remains to describe the reduction from DIST-ICOR. Let $I = (\mathcal{C}, M)$ be an instance of DIST-ICOR with chunks $C_1, \dots, C_\ell$. By replacing each number with its rank among all occurring numbers, we may assume without loss of generality, that the numbers in the sequence are $1, \dots, \sum_{i=1}^{\ell} |C_i| =: L$.

We construct an instance $I' = (\delta_G, K)$ of CIRCULAR UNTANGLING as follows; see Fig. 3a. We create vertices $v_1, \dots, v_L$ and an additional vertex v_0. For each chunk C_i, we create a cycle K_i that starts at v_0, visits the vertices that correspond to the elements of C_i in the given order, and then returns to v_0. That is, G consists of ℓ cycles that are joined by the cut-vertex v_0. The drawing δ_G is obtained by placing the vertices in the clockwise order $\sigma_G = v_0, v_1, v_2, \dots, v_L$ on the unit circle $\mathcal{O}$. Finally, we set $K := L - M$. Clearly, I' can be constructed from I in polynomial time. It remains to prove the following.

Lemma 2.6. *I is a yes-instance of* DIST-ICOR *if and only if I' is a yes-instance of* CIRCULAR UNTANGLING.

Proof. Observe that, since in δ_G the vertices are ordered clockwise according to their numbering, the problem of untangling with at most $L - M$ vertex moves is equivalent to finding a planar circular drawing of G whose clockwise ordering contains an increasing subsequence of at least M vertices, which can then be kept fixed; see Fig. 3b.

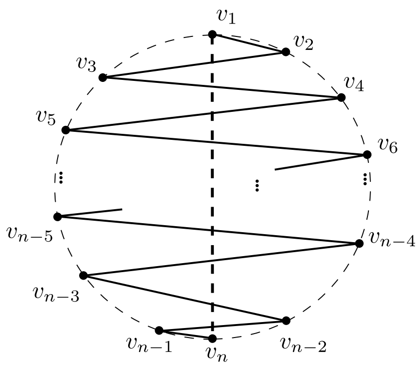


Fig. 4. An almost-planar drawing δ_G with $\mathrm{shift}^\circ(\delta_G) = \frac{n}{2} - 1$.

Since all of the cycles of G are joined at the vertex v_0, the vertices of each cycle K_i are consecutive in every planar circular drawing of G, and the order of its vertices is the order along K_i, i.e., it is fixed up to a reversal. Hence the choice of a circular drawing whose clockwise ordering contains an increasing subsequence of at least M vertices directly corresponds to a permutation and reversals of the chunks C_i. □

3. A tight upper bound of the circular shifting number

In this section, we investigate an upper bound of the circular shifting number and prove the following theorem.

Theorem 3.1. *For every drawing δ_G of an n-vertex outerplanar graph G ($n \geq 3$), we have $\mathrm{shift}^\circ(\delta_G) \leq n - \lfloor\sqrt{n-2}\rfloor - 2$, and this bound is tight.*

Proof. To prove the tightness of the upper bound, we present an untangling that fixes at least $\lfloor\sqrt{n-2}\rfloor + 2$ vertices in the following. Let $G = (V, E)$ be an n-vertex outerplanar graph with a circular drawing δ_G of G. Let δ_G^U be a planar circular drawing of G. We number the vertices of G as $v_1, \ldots, v_n$ in clockwise order according to their occurrence in δ_G^U. Now we untangle δ_G by moving vertices such that the vertices are ordered as $v_1, \ldots, v_n$ clockwise or counterclockwise. To do this with a minimum number of vertex moves is equivalent to finding a longest increasing or decreasing subsequence of the ordering of the vertices in δ_G, which can be fixed during the transformation. The claimed bound follows from the following Erdős-Szekeres Theorem for cyclic permutations.

Theorem 3.2 *([33]). For any two positive integers s, r, any cyclic sequence of $n \geq sr + 2$ distinct real numbers has an increasing cyclic subsequence of $s + 2$ terms or a decreasing cyclic subsequence of $r + 2$ terms, and this bound is tight.*

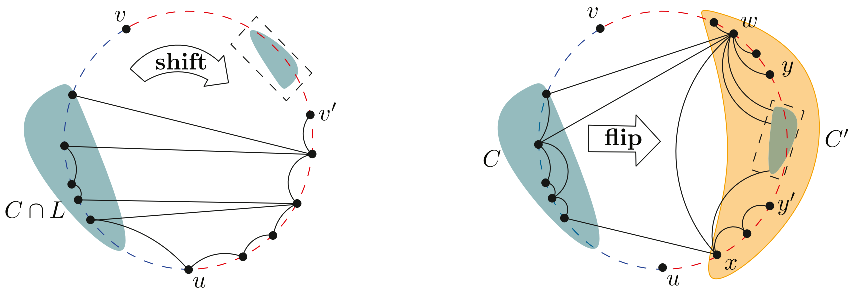
Moreover, observe that for cycles, the circular order of the vertices of a planar drawing is unique up to reversal, and therefore untangling a drawing of a cycle with a minimum number of moves is equivalent to determining a longest increasing or decreasing subsequence in the fixed cyclic ordering determined by the cycle. Hence, a tight example can be obtained from a tight example for the above theorem. □

4. A tight upper bound for almost-planar drawings

In this section, we discuss the upper bound of untangling an almost-planar circular drawing. We show that only almost half of the vertices required to be moved to untangle an almost-planar drawing. Let $G = (V, E)$ be an outerplanar graph and let δ_G be an almost-planar circular drawing of G. In this section, we present an untangling for such almost-planar circular drawings that provides a tight upper bound of $\lfloor\frac{n}{2}\rfloor - 1$ on $\mathrm{shift}^\circ(\delta_G)$.

Theorem 4.1. *For every almost-planar drawing δ_G of an n-vertex outerplanar graph G ($n \geq 4$), we have $\mathrm{shift}^\circ(\delta_G) \leq \lfloor\frac{n}{2}\rfloor - 1$, and this bound is tight.*

To see that the bound is tight, let $n \geq 4$ and let G be the cycle on vertices $v_1, \ldots, v_n, v_1$ (in this order). We first consider the case that n is an even number. Let δ_G be a drawing inducing the clockwise ordering $v_2, \ldots, v_{2i} \ldots, v_n, v_{n-1}, \ldots, v_{2i+1}, \ldots, v_1$; see Fig. 4. We claim that $\mathrm{shift}^\circ(\delta_G) \geq \frac{n}{2} - 1$. Clearly, the clockwise circular ordering induced on the vertices of G in a crossing-free circular drawing is either $v_1, v_2, \ldots, v_n$ or its reverse. Assume that it is $v_1, v_2, \ldots, v_n$; the other case is symmetric. In δ_G, the $\frac{n}{2}$ odd-index vertices $v_1, \ldots, v_{2i+1} \ldots, v_{n-1}$ and v_n are ordered counterclockwise. Hence, to obtain a clockwise ordering, we need to move all but two of these vertices. Thus, at least $\frac{n}{2} - 1$ vertices in total are required to move.



(a) Case 1 (b) Case 2.2 (non-connecting component)

Fig. 5. Moving a left component, keeping/reversing the clockwise ordering of its vertices.

In the case that n is an odd number, the $\lfloor \frac{n}{2} \rfloor + 1$ odd-index vertices $v_1, \ldots, v_{2i+1} \ldots, v_n$ are ordered counterclockwise and thus all but two of them are required to move.

The remainder of this section is devoted to proving the upper bound. Let $e = uv$ be the edge of δ_G that contains all of the crossings, and let $G_o = G - e$ and δ_{G_o} be the planar circular drawing of G_o by removing the edge e from δ_G. The edge uv partitions the vertices in $V \setminus \{u, v\}$ into the sets L and R that lie on the left and right side of the edge uv (directed from u to v).

Theorem 4.2. *Let δ_G be an almost-planar drawing of an outerplanar graph G. A planar circular drawing of G can be obtained from δ_G by moving only vertices of L or only vertices of R to the other side of e in δ_G and fixing all remaining vertices. The untangling moves only* $\min\{|L|, |R|\}$ *vertices and can be computed in linear time.*

This immediately implies the upper bound from Theorem 4.1, since $|L \cup R| = n - 2$, and therefore $\min\{|L|, |R|\} \leq \lfloor \frac{n}{2} \rfloor - 1$. To prove Theorem 4.2, we distinguish different cases based on the connectivity of u and v in G_o.

Case 1: u, v are not connected in G_o. Consider a connected component C of G_o that contains vertices from L and from R.

Proposition 4.3. *Suppose u, v are not connected in G_o. Let C be a connected component of G_o that contains vertices from L and from R. It is possible to obtain a new almost-planar drawing δ'_G of G from δ_G by moving only the vertices of $C \cap L$ (resp. $C \cap R$) such that C lies entirely on the right (resp. left) of uv in δ'_G.*

Proof. In the following, we present how to move vertices of $C \cap L$ to the right side; the other case moving $C \cap R$ to the left side is symmetric. Since u, v are not connected in G_o, C contains at most one of u, v. Without loss of generality, we assume that $v \notin C$; see Fig. 5a. Let v' be the first clockwise vertex after v that lies in C. Let δ'_G be the drawing obtained from δ_G by moving the vertices of $C \cap L$ clockwise just before v' without changing their clockwise ordering. Observe that this movement removes all crossings of e with C. The choice of v' ensures that there exists no edge of C such that its endpoints alternate with endpoints of any edge in $V \setminus C$. Finally, the vertices of C maintain their clockwise order. This shows that no new crossings are introduced, and the crossings between e and C are removed. □

By applying Proposition 4.3 to each connected component of G_o that contains vertices from L and from R, we obtain a planar circular drawing of G.

Case 2: u, v are connected in G_o. Let C be the connected component of G_o that contains both vertices u and v. Note that if C' is another connected component of G_o, then it must lie entirely to the left or entirely to the right of the edge e. Here, we ignore such components as they never need to be moved. We may hence assume that G_o is connected.

Case 2.1: u, v are 2-connected in G_o. We claim that in this case δ_G is already planar.

Proposition 4.4. *If u and v are 2-connected in G_o, then δ_G is planar.*

Proof. If vertices $u, v \in V$ are 2-connected in G_o, then G_o contains a cycle C that includes both u and v. In δ_{G_o}, this cycle is drawn as a closed curve. Any edge of δ_{G_o} that intersects the interior region of this closed curve therefore has both endpoints in C. If there exists an edge $e' = xy$ of G_o that intersects $e = uv$, then contracting the four subpaths of C connecting each of $\{x, y\}$ to each of $\{u, v\}$ yields a K_4-minor in G, which contradicts the outerplanarity of G. □

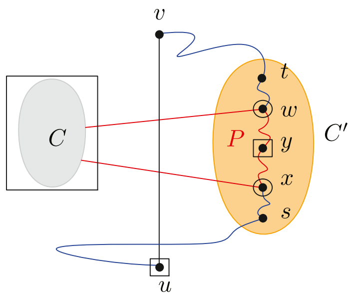


Fig. 6. The $K_{2,3}$-minors we use in the proof of Lemma 4.7.

Case 2.2: u, v **are connected but not 2-connected in** G_o**.** In this case G_o contains at least one cut-vertex that separates u and v. Notice that each path from u to v visits all such cut-vertices between u and v in the same order. Let f and l be the first and the last cut-vertex on any uv-path. Additionally, add u to the set of L, R that contains f and likewise add v to the set of L, R that contains l. Let X denote the set of edges of G_o that have one endpoint in L and the other in R. Each connected component of $G_o - X$ is either a subset of L or a subset of R, which are called *left* and *right components*, respectively. We call a component of $G_o - X$ *connecting* if it contains either u or v, or removing it from G_o disconnects u and v. For a left component C_L and a right component C_R, we denote by $E(C_L, C_R)$ the set of the edges of G_o that connect a vertex from C_L to a vertex in C_R. We get the following observations.

Observation 4.5. For any edge that connects a left and a right component, at least one of the components must be connecting.

Observation 4.6. If P is a path in a left (right) component C connecting two vertices x and y, then it contains all vertices of C that are adjacent to a vertex of a right (left) component and lie between x and y on the left (right) side of uv.

Lemma 4.7. *Every non-connecting component C of $G_o - X$ is adjacent in G_o to exactly one component C' of $G_o - X$. Moreover, C' is connecting, there are at most two vertices in C' that are incident to edges in $E(C, C')$, and if there are two such vertices $w, x \in C'$, then they are adjacent in G_o and removing the edge wx disconnects C'.*

Proof. Without loss of generality, we assume that C is a left component. Since C is non-connecting, any component adjacent to it must be connecting. Moreover, if there are two distinct such components, they lie on the right side of the edge uv. Then either there is a path on the right side that connects them (but then they are not distinct), or removing C disconnects these components, and therefore u and v, contradicting the assumption that C is a non-connecting component. Therefore C is adjacent to exactly one other component C', which must be a right connecting component.

Let w and x be the first and the last vertex in C' that are adjacent to vertices in C when sweeping the vertices of G clockwise in δ_G starting at v; see Fig. 6. The lemma holds trivially if $w = x$. Suppose $w \neq x$. Next we show that the two vertices w and x are adjacent in G_o and that the edge wx is a bridge of C'. Let P be an arbitrary path from w to x in C'. If P contains an internal vertex y, then the path P together with a path from w to x whose internal vertices lie in C forms a cycle, where x and w are not consecutive. Note that at least one of u, v, say u, is not identical to w, x, otherwise, u, v are 2-connected. This cycle, together with disjoint paths from w to v and x to u and the edge uv yields a $K_{2,3}$-minor in G; see Fig. 6. Such paths exist, by the outerplanarity of δ_{G_o} and the fact that C' is connecting, but C is not. Since G is outerplanar, and therefore cannot contain a $K_{2,3}$-minor, this immediately implies that P consists of the single edge wx, which must be a bridge of C' as otherwise there would be a wx-path with an internal vertex. Observation 4.6 implies that w and x are the only vertices of C' that are adjacent to vertices of C. □

Proposition 4.8. *Let C be a left (right) non-connecting component of $G_o - X$. It is always possible to obtain a new almost-planar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side.*

Proof. Without loss of generality, we assume that C is a left component. Since C is non-connecting, then by Lemma 4.7, it is adjacent to at most two vertices on the right side. If there are two such vertices, denote them by w and x such that w occurs before x on a clockwise traversal from v to u. Note that wx is a bridge of a right component C' by Lemma 4.7; see Fig. 5b. Let y be the last vertex that lies in the same component of $C' \setminus \{w, x\}$ as w when traversing vertices clockwise from w to x. If C is adjacent to only one vertex on the right side, then we denote this vertex by y. In both cases, if $y \neq u$ then let y' be the vertex of R that immediately succeeds y in the clockwise direction and if $y = u$ then let y' be the vertex that immediately precedes y in the clockwise direction.

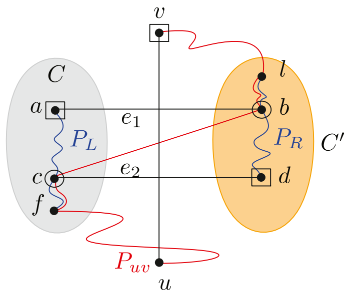


Fig. 7. The $K_{2,3}$-minor we use in the proof of Lemma 4.9.

We obtain δ'_G by moving all vertices of $C \setminus \{u, v\}$ between y and y', reversing their clockwise ordering. Observe that the choice of y and y' guarantees that δ'_G is almost-planar and all crossings lie on uv. □

It remains to deal with the connecting components.

Lemma 4.9. *The connecting component of $G_o - X$ containing u or v is adjacent to at most one connecting component. Every other connecting component is adjacent to exactly two connecting components. Moreover, if C and C' are two adjacent connecting components, then there is a vertex $w \in C \cup C'$ that is incident to all edges in $E(C, C')$.*

Proof. The claims concerning the adjacencies of the connecting components follow from the fact that every uv-path visits all connecting components in the same order. It remains to prove that all edges between two connecting components share a single vertex. If u and v are in one component, then this component is the only connecting component and there is nothing to show.

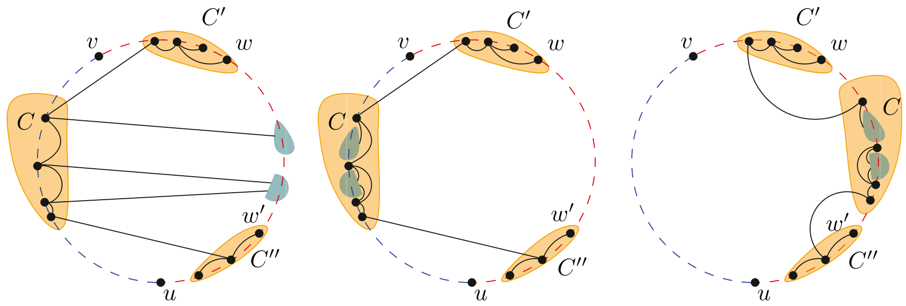
Now let C and C' be adjacent connecting components. We assume without loss of generality, that C is a left and C' is a right component. For the sake of contradiction, assume there exist two edges $e_1, e_2 \in E(C, C')$ that do not share an endpoint. Let $e_1 = ab$ and $e_2 = cd$ where $a, c \in C$ and $b, d \in C'$ such that their clockwise order is a, b, d, c; see Fig. 7. Note that one of u, v is not in the set $\{a, b, c, d\}$. Otherwise, u and v are 2-connected, which contradicts our case assumption. In the following, we assume without loss of generality that a, b, c, d, v are five distinct vertices in G_o. Let P_{uv} be a path from u to v in G_o. Since C and C' are both connecting, P_{uv} contains vertices from both components. When traversing P_{uv} from u to v, let f and l denote the first and the last vertex of $C \cup C'$ that is encountered, respectively. Here, we assume without loss of generality that $f \in C$ and $l \in C'$. Let P_L be a path in C that connects f to a and let P_R be a path in C' that connects d to l. By Observation 4.6, P_L contains c and P_R contains b. For a path P and two vertices x and y of P, let $P[x, y]$ denote the subpath of P from x to y. We then obtain a $K_{2,3}$-minor of G by contracting each of the paths $P_L[c, a]$, $P_R[d, b]$, $vuP_{uv}[u, f]P_L[f, c]$, and $P_R[b, l]P_{uv}[l, v]$ into a single edge. □

By Lemma 4.7 and Lemma 4.9, all vertices of a connecting component of $G_o - X$ can be moved to the other side, thus we get Proposition 4.10, similarly as Proposition 4.8 for non-connecting components.

Proposition 4.10. *Let C be a left (right) connecting component of $G_o - X$. It is possible to obtain a new almost-planar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side of uv.*

Proof. We assume without loss of generality that C is a left connecting component. Now, we determine two vertices w and w' of G such that a right component is a non-connecting component adjacent to C iff it lies between w and w' entirely. If u, v are not in C, by Lemma 4.9, C is adjacent to exactly two right connecting components C', C'' (see Fig. 8a). In the following, we assume that v, C', C'', u are in clockwise order. Let w be the last vertex in C' and w' be the first vertex in C'' when traversing the vertices in δ_G clockwise from v. If C contains both u and v, let w be v and w' be u. If C contains either u or v, by Lemma 4.9, C is adjacent to exactly one right connecting components C'. Assume without loss of generality that $v \in C$. Let w be the last vertex in C' when traversing the vertices in δ_G clockwise and w' be u. Observe that, due to the connectivity of G_o and the outerplanarity of δ_{G_o}, each right component that entirely lies between w and w' is a non-connecting component adjacent to C. Again, we want to only move the component C to the right side between w and w' without introducing any crossings.

For simplicity, we describe the procedure in two phases. In the first phase, we move all of the right non-connecting components connected to C to the left side "temporarily" by the procedure described in the proof of Proposition 4.8 such that the components are merged into C on the left; see Fig. 8b. In the second phase, we move the set $C \setminus \{u, v\}$ (alongside the vertices that are moved in the first phase) to the right side between w and w', reversing their clockwise ordering; see Fig. 8c. For each right component $C^\star$ that is adjacent to C, by Lemma 4.9, there is exactly one vertex shared by edges $E(C, C^\star)$. Thus, there is no crossing on the right side of uv after the second phase. Furthermore, the vertices moved to the



(a) to move C to the right side (b) 1.Phase (c) 2.Phase

Fig. 8. Case 2.2 (connecting-component).

left at the first phase are in the same order as in δ_G after two reversals and they still lie between w and w'. Therefore, we can reach the same order after this two-phase procedure by only moving the vertices in C to the right side accordingly. □

Proposition 4.8 and Proposition 4.10 together imply Theorem 4.2.

In the last part of this section, we consider optimal untangling under the restriction that the positions of u and v are fixed. We denote such untangling as *edge-fixed untangling*.

Theorem 4.11. *Given an almost-planar drawing δ_G of an outerplanar graph G, an edge-fixed untangling of δ_G with the minimum number of vertex moves can be computed in linear time.*

Proof. Let $e = uv$ be the crossed edge of δ_G, let C be a connected component of $G_o = G - e$, and let $L_C = L \cap C$ and $R_C = R \cap C$. Note that every edge-fixed untangling must either move L_C entirely to the right or R_C entirely to the left of edge e. Thus, any edge-fixed untangling must move $\min\{|L_C|, |R_C|\}$ vertices in each component C.

It remains to prove that we can compute such a move sequence with the minimum number of required vertex moves for each component C. If u, v are not connected in G_o, the claim is exactly the same as Proposition 4.3. We now consider the case that u, v are connected in G_o. Let C be the connected component of G_o that contains both u and v. We can always move either L_C or R_C by Proposition 4.8 and Proposition 4.10. Note that any other connected component C' of G_o must lie entirely to the left or entirely to the right of edge e since δ_{G_o} is planar and u, v are connected in G_o. □

5. Untangling almost-planar drawings

Finally, we consider how to untangle an almost-planar circular drawing δ_G of an n-vertex outerplanar graph $G = (V, E)$ with the minimum number of vertex moves. The main result of this section is the following theorem, which we prove by combining the claims of two propositions.

Theorem 5.1. *We can compute a minimum untangling for an almost-planar circular drawing δ_G of an n-vertex outerplanar graph $G = (V, E)$ in $O(n^2)$ time.*

Let $e = uv$ be the edge of δ_G that contains all of the crossings, and let $G_o = G - e$ and δ_{G_o} be the straight-line circular drawing of G_o by removing the edge e from δ_G. The edge uv partitions the vertices in $V \setminus \{u, v\}$ into the sets L and R that lie on the left and right side of the edge uv (directed from u to v). Let C_u and C_v be the connected components of G_o that contain u and v, respectively. Note that $C_u = C_v$ if u, v are in the same connected component of G_o.

Proposition 5.2. *It is always possible to untangle δ_G by moving only the vertices of C_u or only the vertices of C_v.*

Proof. If $C_u = C_v$ the claim is trivially true. So let us consider the case that u and v are not connected in G_o. We describe the untangling by moving C_u entirely as follows; with the same idea, we can untangle δ_G by moving C_v. Let σ_u be the clockwise order of C_u in δ_{G_o}, starting with u. We insert the vertices of C_u in the order σ_u clockwise right after v to obtain a new drawing δ'_{G_o} of G_o. Since C_u was crossing-free before and is placed consecutively on the circle, it remains crossing-free. No other edges have been moved. Furthermore, u and v are now neighbors on the circle, so we can insert the edge uv without crossings and have untangled δ_G. □

It is clear from Proposition 5.2 that we can untangle δ_G by moving all vertices of the smaller of the two connected components C_u and C_v, so we obtain $\text{shift}^\circ(\delta_G) \leq \min\{|C_u|, |C_v|\}$. Assuming that the untangling from Proposition 5.2 is not

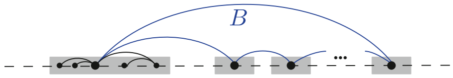


Fig. 9. A 2-connected component B (in blue) and its attachments (gray boxes) in a planar circular drawing.

minimal, we need to find a minimum untangling with $\text{shift}^\circ(\delta_G) < \min\{|C_u|, |C_v|\}$ vertex moves. Thus it remains to consider the case, where some vertices of C_u and some vertices of C_v are not moved; we call such unmoved vertices *fixed* vertices and an untangling with fixed vertices in both components C_u and C_v a *component-fixed untangling*. Then Theorem 5.1 is obtained by choosing the untangling with fewer vertex moves from the ones provided by Propositions 5.2 and 5.3.

Proposition 5.3. *A component-fixed untangling U with the minimum number of vertex moves can be found in $O(n^2)$ time.*

The remainder of this section is devoted to prove Proposition 5.3. We distinguish between two cases based on whether u, v are connected in G_o or not. In each case, we present an untangling that runs in $O(n^2)$ time and reports an optimal component-fixed untangling.

We introduce some notions and provide basic observations. Let G be a connected outerplanar graph. Let B be a 2-connected component of G and $E(B)$ the set of edges in B. Since G is connected and B is 2-connected, each connected component of $G - E(B)$ contains exactly one vertex in B. Given a vertex b in B, let C_b be the connected component of $G - E(B)$ that contains b. We denote C_b as the *attachment* of the 2-connected component B at the vertex b. Note that C_b can consist of the single vertex b.

Recall that every 2-connected outerplanar graph has a unique Hamiltonian cycle [29]. Let $H(B)$ be the cyclic vertex ordering of B in the order of its Hamiltonian cycle. We get Observation 5.4; see Fig. 9.

Observation 5.4. Let δ_G be a planar circular drawing of a 2-connected outerplanar graph G and B be a 2-connected component of G. Then, the clockwise cyclic vertex ordering of B in δ_G is either $H(B)$ or its reverse. Furthermore, for each attachment of B, its vertices appear consecutively on the circle in δ_G.

Given a connected outerplanar graph G, a 2-connected component B of G, and a circular drawing δ_G of G, we say a sequence S of vertex moves of G is *canonical*, with respect to B, if in the drawing obtained by applying S to δ_G, the clockwise cyclic vertex ordering of each attachment of B remains unchanged. Now we are ready to show that an optimal component-fixed untangling with the restriction that fixed vertices exist in both of C_u and C_v can be found in $O(n^2)$ time; see Proposition 5.3.

Case 1: u and v are connected in G_o. Let C be a connected component of G_o that does not contain u, v. We claim now that C must lie entirely on one side of uv in δ_G. Otherwise, let P be a path of δ_{G_o} that connects u and v. Then there would exist crossings between edges of P and edges of C in δ_{G_o} which contradicts the fact that δ_{G_o} has no crossings. Thus, we can ignore such components as they do not need to be involved in an untangling. Hence, we may assume G_o is a connected graph. If u and v are 2-connected in G_o, then δ_G is already outerplanar; see Proposition 4.4. Now we consider the case that u and v are connected, but not 2-connected in G_o. Note that u, v are 2-connected in G. Let B be the 2-connected component of G that contains u, v. We prove that each component-fixed untangling U can be transformed into a canonical untangling with smaller or the same number of vertex moves; see Lemma 5.5. Thus, we restrict our attention to canonical untanglings. Let $H(B) = b_1, \dots b_k$ be the cyclic vertex ordering of the Hamiltonian cycle of B. Let A_i be the attachment of B at the vertex b_i and let $\sigma(A_i)$ be the clockwise vertex ordering of A_i in δ_G for $i \in \{1, \dots, k\}$. We consider an optimal canonical component-fixed untangling U^o which orders B clockwise as $H(B)$. Let δ''_G be the outerplanar drawing obtained by applying U^o to δ_G. Then the clockwise vertex ordering of δ''_G is exactly the concatenation of $\sigma(A_1), \sigma(A_2), \dots, \sigma(A_k)$. Given δ''_G, an optimal untangling transforming δ_G to δ''_G can be computed in $O(n^2)$ time; see [24]. Analogously, we obtain an optimal component-fixed untangling U^r which orders B counterclockwise as $H(B)$. From the two untanglings U^o and U^r, we report the one which moves less vertices as the optimal component-fixed untangling.

Lemma 5.5. *Let B be the 2-connected component of G that contains u, v. Every component-fixed untangling U of δ_G can be transformed into a canonical vertex move sequence U^c (with respect to B) that untangles δ_G. Furthermore, the number of vertex moves in U^c is not greater than the number of vertex moves in U.*

Proof. Given a component-fixed untangling U of δ_G, let δ_G^U be the drawing obtained after applying U on δ_G. In δ_G^U, the cyclic vertex ordering of B (clockwise or counterclockwise) must correspond to its Hamiltonian cycle ordering $H(B)$. Furthermore, the vertices of each attachment of B appear consecutively in δ_G^U, including one vertex of B; see Observation 5.4. Let $A_1, \dots, A_k$ be the attachments of B in G (indexed in clockwise order as in δ_G^U) and let $\sigma(A_i)$ be the clockwise vertex ordering of A_i in δ_G for $i \in \{1 \dots k\}$. Now consider the vertex ordering $\sigma'_G = (\sigma(A_1), \cdots, \sigma(A_k))$ and let δ'_G be an arbitrary circular drawing of G where the vertices are ordered as σ'_G. Note that the vertex ordering of each attachment is $\sigma(A_i)$ in

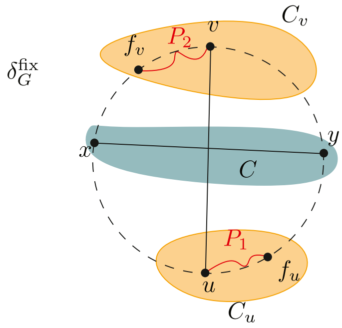


Fig. 10. An example illustration for the proof of Lemma 5.7.

δ'_G as in the almost-planar drawing δ_G, thus each attachment in δ'_G is crossing-free. Moreover, in δ'_G the vertices of B are ordered as in the planar drawing δ_G^U, thus there is no crossing inside B. Overall, δ'_G is a planar circular drawing. Let U^c be the untangling of δ_G with minimum number of vertex moves such that the clockwise vertex ordering of the resulting drawing is σ'_G.

To see that U^c does not move more vertices than U, let σ_G and σ_G^U be the clockwise vertex orderings of δ_G and δ_G^U, respectively. Let $LCS(A, B)$ be the longest common sequence between two cyclic ordering A and B. We can observe that $|LCS(\sigma_G, \sigma_G^U)| \le |LCS(\sigma_G, \sigma'_G)|$, since any common subsequence of σ_G and σ_G^U is a subsequence of σ'_G. $\square$

Case 2: u and v are not connected in G_o. Note that a connected component of G_o that lies entirely on one side of uv in δ_G can be ignored, since there is no need to move any vertices in such components. We can assume that each connected component C of G_o either contains vertices from L and also vertices from R or C contains either u or v.

Observation 5.6. We can assume that vertices of C_u (resp. C_v) lie consecutively on the cycle in δ_{G_o}.

The first step of our untangling U deals with the connected components of G_o that neither contain u nor v. Let U^{fix} be an arbitrary component-fixed untangling of δ_G, and let δ_G^{fix} be the outerplanar drawing of G obtained from δ_G by applying U^{fix}.

Lemma 5.7. *Let C be a connected component of G_o that contains neither u nor v. Let f_u, f_v be two vertices in C_u and C_v, respectively, which are fixed in δ_G^{fix}. Then, C must lie entirely on one side of $f_u f_v$ in δ_G^{fix}.*

Proof. In the graph G, due to the definition of f_u and f_v, there exists a path P_1 in C_u connecting f_u to u, and a path P_2 in C_v connecting v to f_v; see Fig. 10. Then, the path $P = P_1 u v P_2$ in G connects f_u to f_v. In δ_G^{fix}, suppose that the connected component C is not entirely on one side of $f_u f_v$, it implies that at least one edge xy in C has endpoints x, y alternate with f_u, f_v in the clockwise ordering of δ_G^{fix} and then has crossings with P. It contradicts the outerplanarity of the drawing δ_G^{fix}. $\square$

Now let C (marked as green in Fig. 10) be a connected component of G that contains neither u nor v. Let f_u, f_v be two vertices in C_u and C_v, respectively, which are fixed in δ_G^{fix}. The vertices f_u and f_v partition the vertices of C in the drawing δ_G into two sets L_C and R_C that are encountered clockwise and counter-clockwise from f_u to f_v in δ_G, respectively. Observe that, $L_C = L \cap C$ and $R_C = R \cap C$; see Observation 5.6. Let $m(C) = \min\{|L \cap C|, |R \cap C|\}$. By Lemma 5.7, $m(C)$ is a lower bound of the moved vertices in C in a component-fixed untangling. On the other hand, by Proposition 4.3, we can move $m(C)$ vertices of C such that C lies entirely on one side of uv. In the first step of our untangling U, we repeat this step for each component not containing u or v. After that, an almost-planar drawing of G remains that has already each component except C_u and C_v placed entirely on one side of uv. We can ignore such components from now on since they never need to be moved again.

Now we assume that G_o has exactly two connected components, namely C_u and C_v. Consider an arbitrary outerplanar drawing δ'_G of G. Let σ be the circular ordering of vertices in δ'_G encountered clockwise. Observe that, in σ, the vertices of C_u (resp. C_v) are in a consecutive subsequence $\sigma(C_u)$ (resp. $\sigma(C_v)$). Otherwise, alternating vertices of two connected components would introduce crossings (see Fig. 11).

Given an edge e' in C_v, we say e' *covers* v if the endpoints of e alternate with u and v in δ_{G_o}. Note that there is no edge covering v in $\sigma(C_v)$. Otherwise, such an edge would cross with the edge uv. Therefore, in a valid untangling of δ_G, it is necessary to move vertices of C_v in δ_G such that no crossing is introduced in C_v and v is not covered by any edges in C_v. Similarly, the same claim holds also for C_u. We call such vertex moves *vertex unwrapping*. In the following, we consider how to find a valid unwrapping of v with the minimum number of vertex moves. The same operation will be also applied

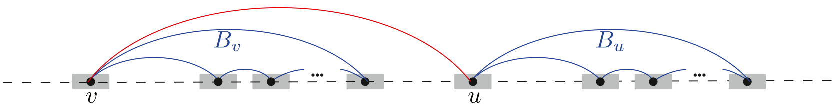


Fig. 11. In any planar circular drawing of G, the vertices of C_u (resp. C_v) appear consecutively. Let B_i be any 2-connected component of G_o containing the vertex i, where $i \in \{u, v\}$. Then the vertex i must be an extreme vertex in B_i.

to C_u. Observe that, once u, v are both unwrapped, adding the edge e into the drawing does not introduce any crossings. The combination of these two unwrappings makes an optimal untangling.

Observation 5.8. There exists a 2-connected component B of C_v such that B contains v and no edge in the attachment of v (with respect to B) covers v in δ_{G_o}.

Observation 5.8 holds because either no 2-connected component B containing v contains an edge covering v in δ_G, in which case v is already unwrapped and the statement is true for any such B. Or some 2-connected component B does contain an edge covering v in δ_G, but then the attachment of v in B cannot cover v due to planarity of δ_{G_o}.

Here, we also consider canonical sequences of vertex moves and get the following Lemma 5.9. Its proof is quite similar to the proof of Lemma 5.5, which concerns canonical untanglings.

Lemma 5.9. *Let B be a 2-connected component of C_v that contains v such that the attachment of v (with respect to B) contains no edge covering v. Each unwrapping W of v can be transformed into a canonical sequence of vertex moves W^c, which unwraps v. Furthermore, the number of vertex moves in W^c is not greater than the number of vertex moves in the original unwrapping W.*

Proof. Given an unwrapping W of v, let δ_G^W be the drawing obtained after applying W on δ_G. In δ_G^W, the cyclic vertex ordering of B (clockwise or counterclockwise) must correspond to its Hamiltonian cycle ordering $H(B)$. Furthermore, the vertices of each attachment of B appear consecutively in δ_G^W, including one vertex of B; see Observation 5.4. Let $A_1, \ldots A_k$ be the attachments of B in C_v (in this clockwise order in δ_G^W), let $\sigma(A_i)$ be the clockwise vertex ordering of A_i in δ_G for $i \in \{1 \ldots k\}$. Consider the clockwise vertex ordering σ'_G where the vertices of $B \cup C_u$ are ordered as in δ_G^W. Furthermore, for each attachment A_i the vertices of A_i appear consecutively in the clockwise ordering $\sigma(A_i)$. Let δ'_G be an arbitrary circular drawing of G where the vertices are ordered as σ'_G. Note that the vertex ordering of each attachment of B is $\sigma(A_i)$ in δ'_G as in the almost-planar drawing δ_G, thus each attachment in δ'_G is crossing-free. Moreover, in δ'_G the vertices of B are ordered as in the planar drawing δ_G^W, thus there is no crossing inside B. Overall, the vertex v is unwrapped in δ'_G. It remains to prove that the canonical unwrapping W^c, which transforms δ_G to δ'_G, moves less than or equally many vertices of C_v as W. This follows from the construction of δ'_G, because each common subsequence of δ_G and δ_G^W is also a subsequence of δ'_G. □

By Lemma 5.9, we restrict our attention to canonical unwrappings. We first find a 2-connected component B_v of C_v containing v such that no edge in the attachment (with respect to B_v) of v covers v in δ_G. To find such a component B_v we can go through all 2-connected components containing v, which correspond to cycles or edges containing v in δ_{G_o}. It takes linear time. We then consider the two possible canonical unwrappings of v, which respectively order vertices of B clockwise along $H(B)$ or its reversal, and compute the corresponding resulting clockwise vertex ordering σ_v and σ_v^{rev} of C_v. With the same idea, we get the clockwise vertex orderings σ_u and σ_u^{rev} of C_u by the canonical unwrappings of u. We then get the four optimal unwrappings, each of them transforming δ_G to one of the vertex orderings $(\sigma_v\sigma_u)$, $(\sigma_v^{rev}\sigma_u)$, $(\sigma_v\sigma_u^{rev})$ and $(\sigma_v^{rev}\sigma_u^{rev})$. Such optimal unwrappings can be computed in $O(n^2)$ time; see [24]. We report the one that moves the minimum number of vertices as an optimal component-fixed untangling.

6. Conclusion and outlook

We introduced and investigated the problem of untangling non-planar circular drawings. First from the computational side, we demonstrated the NP-hardness of the problem Circular Untangling. Second, we studied the almost-planar circular drawings, where all crossings involve a single edge. We gave a tight upper bound of $\lfloor \frac{n}{2} \rfloor - 1$ on the shift number and an $O(n^2)$-time algorithm to compute it. Open problems for future work include: (i) The parameterized complexity of computing the circular shifting, e.g., with respect to the number of crossings or the number of connected components, (ii) Generalization of our results for almost-planar drawings, and (iii) Investigation of minimum untangling by other elementary moves such as swapping vertex pairs or moving larger chunks of vertices.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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