Untangling Circular Drawings: Algorithms and Complexity

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Abstract

We consider the problem of untangling a given (non-planar) straight-line circular drawing δ_G of an outerplanar graph $G = (V, E)$ into a planar straight-line circular drawing by shifting a minimum number of vertices to a new position on the circle. For an outerplanar graph G, it is clear that such a crossing-free circular drawing always exists and we define the *circular shifting number* $\text{shift}^\circ(\delta_G)$ as the minimum number of vertices that are required to be shifted in order to resolve all crossings of δ_G. We show that the problem Circular Untangling, asking whether $\text{shift}^\circ(\delta_G) \le K$ for a given integer K, is NP-complete. For n-vertex outerplanar graphs, we obtain a tight upper bound of $\text{shift}^\circ(\delta_G) \le n - \lfloor\sqrt{n-2}\rfloor - 2$. Based on these results we study Circular Untangling for *almost-planar* circular drawings, in which a single edge is involved in all the crossings. In this case we provide a tight upper bound $\text{shift}^\circ(\delta_G) \le \lfloor \frac{n}{2} \rfloor - 1$, and present a constructive polynomial-time algorithm to compute the circular shifting number of almost-planar drawings.

2012 ACM Subject Classification Theory of computation → Computational geometry; Human-centered computing → Graph drawings; Mathematics of computing → Permutations and combinations; Theory of computation → Problems, reductions and completeness

Keywords and phrases graph drawing, straight-line drawing, outerplanarity, NP-hardness, untangling

Acknowledgements G.L. and M.N. acknowledge support by the Austrian Science Fund (FWF) under grant P 31119. I.R. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Ru 1903/3-1.

1 Introduction

The family of outerplanar graphs, i.e., the graphs that admit a planar drawing where all vertices are incident to the outer face, is an important subclass of planar graphs and exhibits interesting properties in algorithm design, e.g., they have treewidth at most 2. Being defined by the existence of a certain type of drawing, outerplanar graphs are a fundamental topic in the field of graph drawing and information visualization; they are relevant to circular graph drawing [27] and book embedding [3, 5]. Several aspects of outerplanar graphs have been studied over the years, e.g., characterization [8, 28, 13], recognition [30, 1], and drawing [14, 20, 26]. Moreover, outerplanar graphs and their drawings have been applied to various scientific fields, e.g., network routing [15], VLSI design [9], and biological data modeling and visualization [19, 31].

In this paper we study the untangling problem for non-planar circular drawings of outerplanar graphs, i.e., we are interested in restoring the planarity property of a straight-line

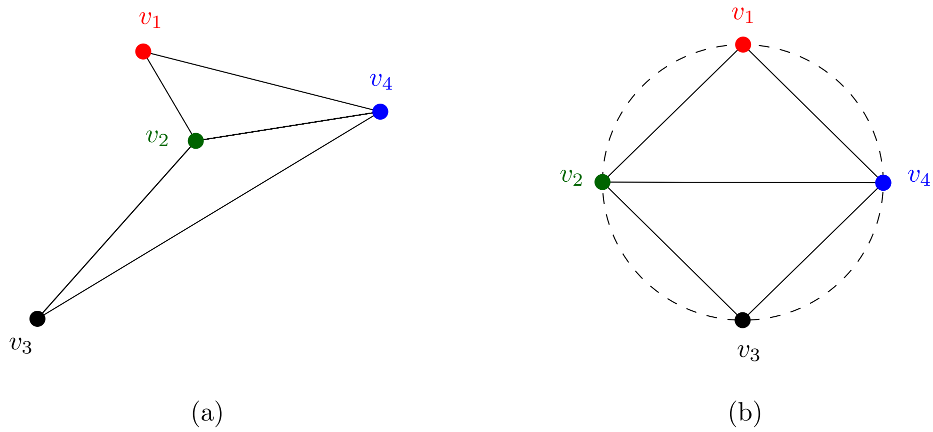


Figure 1 morphing an outerplanar drawing (a) into a circular drawing (b)

circular drawing with a minimum number of vertex shifts. Similar untangling concepts have been used previously for eliminating edge crossings in non-planar drawings of planar graphs [17]. More precisely, let $G = (V, E)$ be an n-vertex outerplanar graph and let δ_G be an outerplanar drawing of G, which can be described combinatorially as the (cyclic) order $\sigma = (v_1, v_2, \ldots, v_n)$ of V when traversing vertices on the boundary of the outer face counterclockwise. This order σ corresponds to a planar circular drawing by mapping each vertex $v_i \in V$ to the point p_i on the unit circle $\mathcal{O}$ with polar coordinate $p_i = (1, \frac{i}{n} \cdot 2\pi)$ and drawing each edge $(v_i, v_j) \in E$ as the straight-line segment between its endpoints p_i and p_j; see Figure 1. We note that it is sufficient to consider circular drawings since any outerplanar drawing can be transformed into an equivalent circular drawing by morphing the boundary of the outer face to $\mathcal{O}$ and then redrawing the edges as straight segments.

Our untangling problem is further motivated by the problem of maintaining an outerplanar drawing of a *dynamic* outerplanar graph, which is subject to edge or vertex insertions and deletions, while maximizing the visual *stability* of the drawing [21, 22], i.e., the number of vertices that can remain in their current position. Such problems of maintaining drawings with specific properties for dynamic graphs have been studied before [11, 12, 2, 4], but not for the outerplanarity property.

Related Work. The notion of untangling is often used in the literature for a crossing elimination procedure that makes a non-planar drawing of a planar graph crossing-free; see [10, 18, 24, 25]. Given a straight-line drawing δ_G of a planar graph G, the problem to decide if one can untangle δ_G by moving at most K vertices, is known to be NP-hard [17, 29]. Lower bounds on the number of vertices that can remain fixed in an untangling process have also been studied [6, 7, 17]. On the one hand, Bose et al. [6] proved that $\Omega(n^{1/4})$ vertices can remain fixed when untangling a drawing. Cano et al. [7] on the other hand provide a family of drawings, where at most $O(n^{0.4948})$ vertices can remain fixed during untangling. Goaoc et al. [17] proposed an algorithm, which allows at least $\sqrt{(\log n - 1)/\log\log n}$ vertices to remain fixed when untangling a drawing. Given an arbitrary drawing of an n-vertex outerplanar graph, all edge crossings could be eliminated while keeping at least $\sqrt{n/2}$ vertices fixed [25, 17], whereas there exists a drawing δ_G of an n-vertex outerplanar graph G such that at most $\sqrt{n-1}+1$ vertices can stay fixed when untangling δ_G [17]. Note that the untangled drawings in these previous works are planar but not necessary outerplanar. In this paper, we study untangling procedures to obtain an outerplanar circular drawing from a non-outerplanar circular drawing. To the best of our knowledge, there are no previous studies about untangling circular drawings.

Preliminaries and Problem Definition. Given a graph $G = (V, E)$, we say two vertices are *2-connected* if they are connected by two internally vertex-disjoint paths. A 2-connected component of G is a maximal set of pairwise 2-connected vertices. Two subsets $A, B \subseteq V$ are *adjacent* if there is an edge $ab \in E$ with $a \in A$ and $b \in B$. A *bridge* (resp. *cut-vertex*) of G is an edge (resp. vertex) whose deletion increases the number of connected components of G.

A drawing of a graph is *planar* if it has no crossings, it is *almost-planar* if there is a single edge that is involved in all crossings, and it is *outerplanar* if it is planar and all vertices are incident to the outer face. A graph $G = (V, E)$ is *outerplanar* if it admits an outerplanar drawing. In addition, a drawing where the vertices lie on a circle and the edges are drawn as straight-line segments is called a *circular drawing*. Every outerplanar graph G admits a planar circular drawing, as one can start with an arbitrary outerplanar drawing δ_G of G and transform the outer face of δ_G to a circle [27]. In this paper, we exclusively work with circular drawings of outerplanar graphs; we thus simply refer to them as drawings.

Given a non-planar circular drawing δ_G of an n-vertex outerplanar graph G where vertices lie on the unit circle $\mathcal{O}$, we can transform the drawing δ_G to a planar circular drawing by moving the vertices on the circle $\mathcal{O}$. Formally, given a circular drawing δ_G, a vertex move operation (or shift) changes the position of one vertex in δ_G to another position on the circle $\mathcal{O}$ [17]. We call a sequence of moving operations that results in a planar circular drawing an *untangling* of δ_G. We say an untangling is *minimum* if the number of vertex moves of this untangling is the minimum over all valid untanglings of δ_G. We define the *circular shifting number* $\mathrm{shift}^\circ(\delta_G)$ of a circular drawing δ_G as the number of vertex moves in a minimum untangling of δ_G. We study the following problems.

▶ **Problem 1** (Circular Untangling (CU)). *Given a circular drawing δ_G of an outerplanar graph G and an integer K, decide if* $\mathrm{shift}^\circ(\delta_G) \leq K$.

▶ **Problem 2** (Minimum Circular Untangling (MinCU)). *Given a circular drawing δ_G of an outerplanar graph G, find an untangling of δ_G with* $\mathrm{shift}^\circ(\delta_G)$ *vertex moves.*

Contributions. In Section 2, we show that Circular Untangling is NP-complete. In Section 3, we provide a tight upper bound of the circular shifting number. We then consider almost-planar drawings. In this case, we provide a tight upper bound on the circular shifting number in Section 4 and design a quadratic-time algorithm to compute a circular untangling with the minimum number of vertex moves in Section 5.

2 Complexity of Circular Untangling

The goal of this section is to prove the following theorem.

▶ **Theorem 3.** Circular Untangling *is* NP*-complete.*

Ultimately, the NP-completeness follows by a reduction from the well-known NP-complete problem 3-Partition [16]. However, we do not give a direct reduction but rather work via an intermediate problem, called Distinct Increasing Chunk Ordering with Reversals that concerns increasing subsequences. A *chunk* C is a sequence $C = (c_i)_{i=1}^n$ of positive integers. For a chunk C, we denote its reversal by C^{-1}. We introduce the following problem.

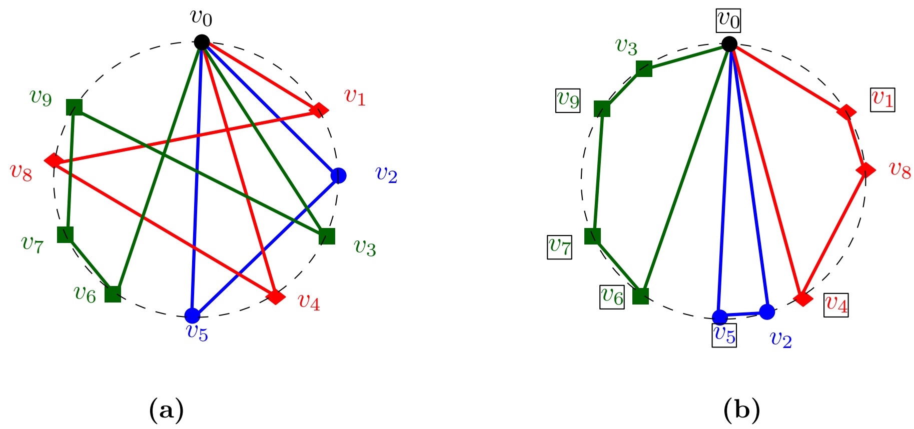


Figure 2 The reduction from DIST-ICOR to CIRCULAR UNTANGLING. (a) The circular drawing δ_G constructed from a DIST-ICOR instance with chunk set $\mathcal{C} = \{C_1 = (1,8,4), C_2 = (2,5), C_3 = (6,7,9,3)\}$. (b) An example drawing obtained by applying an optimum untangling on δ_G. Fixed vertices are marked in $\square$.

▶ **Problem 4** (INCREASING CHUNK ORDERING WITH REVERSALS (ICOR)). *Given a set $\mathcal{C} = \{C_1, \dots, C_\ell\}$ of ℓ* chunks *and a positive integer M, the question is to determine whether a permutation π of $\{1,\dots,\ell\}$ and a function $\varepsilon\colon \{1,\dots,\ell\} \to \{-1,1\}$ exist such that the concatenation $C_{\pi(1)}^{\varepsilon(1)} C_{\pi(2)}^{\varepsilon(2)}, \dots, C_{\pi(\ell)}^{\varepsilon(n)}$ contains a strictly increasing subsequence of length M.*

This problem also comes in a *distinct* variant, denoted DIST-ICOR, where all integers in all input chunks are required to be distinct. In the following, for two problems A and B, we write $A \leq_p B$ if there is a polynomial-time reduction from A to B. It is readily seen that CIRCULAR UNTANGLING lies in NP. Therefore, Theorem 3 follows immediately from the following two reduction lemmas, whose proofs are given in the next two subsections.

▶ **Lemma 5.** *DIST-ICOR $\leq_p$ CIRCULAR UNTANGLING*

▶ **Lemma 6.** *3-PARTITION $\leq_p$ DIST-ICOR*

2.1 Proof of Lemma 5

Let $I = (\mathcal{C}, M)$ be an instance of DIST-ICOR with chunks $C_1, \dots, C_\ell$. By replacing each number with its rank among all occurring numbers, we may assume without loss of generality, that the numbers in the sequence are $1, \dots, \sum_{i=1}^{\ell} |C_i| =: L$.

We construct an instance $I' = (\delta_G, K)$ of CIRCULAR UNTANGLING as follows; see Figure 2a. We create vertices $v_1, \dots, v_L$ and an additional vertex v_0. For each chunk C_i, we create a cycle K_i that starts at v_0, visits the vertices that correspond to the elements of C_i in the given order, and then returns to v_0. That is, G consists of ℓ cycles that are joined by the cut-vertex v_0. The drawing δ_G is obtained by placing the vertices in the clockwise order $\sigma_G = v_0, v_1, v_2, \dots, v_L$ on $\mathcal{O}$. Finally, we set $K := L - M$. Clearly, I' can be constructed from I in polynomial time. It remains to prove the following.

▶ **Lemma 7.** *I is a yes-instance of DIST-ICOR if and only if I' is a yes-instance of CIRCULAR UNTANGLING.*

Proof. Observe that, since in δ_G the vertices are ordered clockwise according to their numbering, the problem of untangling with at most $L - M$ vertex moves is equivalent to finding a planar circular drawing of G whose clockwise ordering contains an increasing subsequence of at least M vertices, which can then be kept fixed; see Figure 2b.

Since all the cycles of G are joined at the vertex v_0, the vertices of each cycle K_i are consecutive in every planar circular drawing of G, and the order of its vertices is the order along K_i, i.e., it is fixed up to reversal. Hence the choice of a circular drawing whose clockwise ordering contains an increasing subsequence of at least M vertices directly corresponds to a permutation and reversal of the chunks C_i. ◀

2.2 Proof of Lemma 6

Let $I = (A, K)$ be an instance of 3-Partition. The input to the 3-Partition problem consists of a multiset $A = \{a_1, \ldots, a_{3m}\}$ of $3m$ positive integers and a positive integer K such that $\frac{K}{4} < a_i < \frac{K}{2}$ for $i = 1, \ldots, 3m$. The question is whether A can be partitioned into m disjoint triplets $T_1, \ldots, T_m$ such that $\sum_{a \in T_j} a = K$ for all $j = 1, \ldots, m$. It is well-known that 3-Partition is strongly NP-complete, i.e., the problem is NP-complete even if the integers in A and K are polynomially bounded in m [16].

Let $I = (A, K)$ with $A = \{a_1, \ldots, a_{3m}\}$ be an instance of 3-Partition. We assume that each number in A is a multiple of $3m$, otherwise, we could multiply each element in A and K by $3m$. We now construct an instance $I' = (C, M)$ of Dist-ICOR in polynomial time.

Construction. We create for each element a_i a corresponding chunk C_i as follows. For two integers a, l, we denote the consecutive integer sequence $(a, a + 1, \ldots, a + l - 1)$ as the *incremental sequence* of length l starting at a. We say that a sequence of integers *crosses* an integer c if it contains both a number that is at most c and a number that is at least $c+1$. Let $X = 3mK$. We take all the incremental sequences of length $a_i + X$ starting at $\alpha \cdot (K + 3X) + \beta \cdot X + \gamma$ for $\alpha \in \{0, \cdots m - 1\}, \beta \in \{0, 1, 2\}$ and $\gamma \in \{1, 2, \cdots, K - a_i\}$. Note that there are at most X such sequences and no such sequence crosses a multiple of $K + 3X$. To construct the chunk C_i, we first build a chunk C'_i with possibly repeating numbers as follows. The chunk C'_i is formed by concatenating these incremental sequences in decreasing order of their starting number; see Figure 3. Observe that, in the figure, a strictly increasing subsequence corresponds to an x-monotone chain of points with positive slopes, whereas a non-increasing subsequence corresponds to an x-monotone chain of points with non-positive slopes.

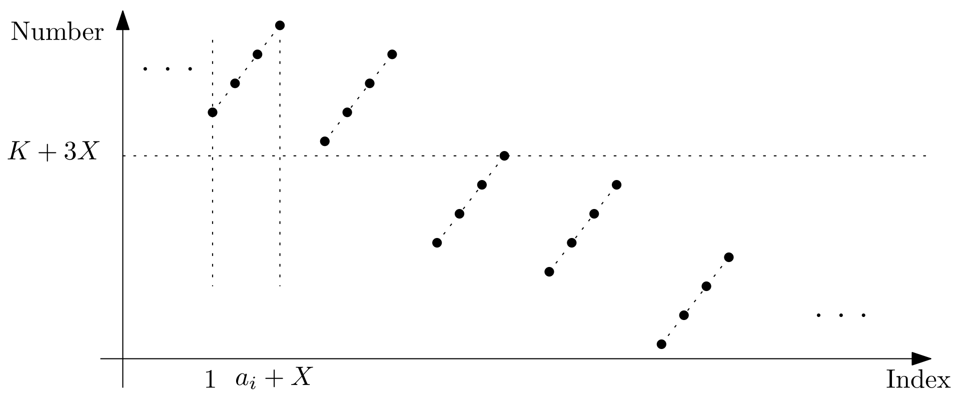


Figure 3 Construction of chunk C_i as a concatenation of incremental sequences of length $a_i + X$ in the decreasing order of their first number.

To make the elements distinct, we introduce strings of numbers, called *words*, which we order lexicographically. We take the concatenation C of chunks $C'_1, C'_2, \cdots C'_{3m}$, then replace the number a at the i-th position by the word $(a, |C| - i)$ for each position i. The chunks $C_1, \ldots, C_{3m}$ are obtained by cutting this modified sequence of words in such a way that $|C_i| = |C'_i|$ for $i = 1, \ldots, 3m$. At the end of the construction, each word is replaced by its rank in a lexicographically increasing ordering of all words that occur in the instances. We obtain an instance $I' = (\mathcal{C}, M)$ of Increasing Chunk Ordering with Reversals by setting $\mathcal{C} = \{C_1, \ldots, C_{3m}\}$ and $M := m(K + 3X)$. For simplicity, we use the construction with words in the following.

For a sequence of words with two entries, we call the sequence obtained by keeping only the first entry of each word, its *projection*. Note that the projection of C_i is C'_i.

▶ **Lemma 8.** *The chunks $C_1, \ldots, C_{3m}$ have the following properties.*

(i) *For every strictly increasing subsequence of C_i, its projection is a strictly increasing sequence.*

(ii) *No projection of a strictly increasing subsequence of C_i crosses a multiple of $K + 3X$.*

(iii) *For $\alpha \in \{0, \cdots m-1\}, \beta \in \{0, 1, 2\}$ and $\gamma \in \{1, 2, \cdots, K - a_i\}$, there exists a subsequence of C_i whose projection is the incremental sequence of length $a_i + X$ starting at $\alpha \cdot (K + 3X) + \beta \cdot X + \gamma$.*

(iv) *Every strictly increasing subsequence of C_i has length at most $a_i + X$.*

(v) *Every strictly increasing subsequence of C_i^{-1} has length at most X.*

Proof. Since the sequence obtained by keeping the second entry of each word of C_i is strictly decreasing, we get Property (i). Property (ii) and Property (iii) follow directly from the construction of C'_i.

To see the Property (iv), consider an arbitrary strictly increasing subsequence s of C'_i. Recall that C'_i is the concatenation of incremental sequences of length $a_i + X$ in decreasing order of their starting number. Given an index $j \in \{1, \cdots a_i + X\}$, we claim that s contains the j-th element of at most one incremental subsequence of C'_i. Otherwise, suppose there are two incremental subsequences of C'_i whose j-th element is in s, then these two numbers are ordered decreasingly, contradicting the assumption that s is strictly increasing. Consequently, the length of strictly increasing subsequence of C'_i is bounded by $a_i + X$.

For Property (v), consider a strictly increasing subsequence of C_i^{-1}. It corresponds to a strictly decreasing subsequence s of C_i, and its projection is a non-increasing subsequence s' of C'_i. Note that s' contains at most one element of each incremental sequence of C'_i, and C'_i is the concatenation of at most X incremental sequences. Therefore, the length of s' is at most X. ◀

▶ **Lemma 9.** *I' is a yes-instance of* Dist-ICOR *if and only if I is a yes-instance of* 3-Partition.

Proof. Assume there is a partition of the elements of A into m triplets, each of which sums to K. We arbitrarily order these triples, and within each triplet, we order the elements according to their index. This defines a total ordering on the elements, and therefore on the chunks. Let $T_i = \{a_x, a_y, a_z\}$ with $x < y < z$ be the ith triplet and let C_x, C_y, C_z be the corresponding chunks. By Property (iii), C_x, C_y, and C_z contain respectively three subsequences whose projections are the incremental sequences of length $a_x + X$, $a_y + X$, and $a_z + X$ starting at $(i-1)(K + 3X) + 1, (i-1)(K + 3X) + X + a_x + 1$, and $(i-1)(K + 3X) + 2X + a_x + a_y + 1$. Concatenating these subsequences for all chunks hence gives an increasing subsequence whose projection the sequence $1, \cdots, m(K + 3X)$.

Conversely, assume that there is a chunk ordering that contains a strictly increasing subsequence S of length $m(K+3X)$. By Property (iv) and Property (v), each chunk C_i or its reversal can contribute a subsequence of at most a_i+X elements, therefore each chunk C_i or its reversal must contribute an increasing subsequence of length a_i+X. Moreover, reversing C_i only provides a shorter increasing subsequence than a_i+X, thus no C_i is reversed. We cut the sequence S into m consecutive sequences $S_1, S_2 \ldots S_m$, called *partition cells* of S, such that the projection of S_i consists of numbers in $\{(i-1)(K+3X)+1, \cdots, i(K+3X)\}$. By Property (ii), the projection of every strictly increasing subsequence inside a chunk does not cross a multiple of $K+3X$, thus each chunk contributes to exactly one partition cell. We claim the following:

▷ Claim 10. Each partition cell has length $K+3X$.

We first show how the proof of the lemma can be derived from the claim. Since the length of each cell is $K+3X$, exactly three chunks contribute to each cell. Each such triplet of chunks then corresponds to a triplet of A whose sum is K. Together, these triplets define a solution of the instance I of 3-Partition.

It remains to prove the claim. Consider a partition cell S_i consisting of numbers from n chunks. Then S_i is the concatenation of subsequences $S_{i,1}, S_{i,2}, \ldots, S_{i,n}$, $n \leq 3m$, each of which is contributed by a different chunk. Since the projection of S_i is a non-decreasing sequence consisting of numbers in $\{(i-1)(K+3X)+1, \cdots, i(K+3X)\}$ and by Property (i), the projection of each $S_{i,j}$ is a strictly increasing sequence, it follows that non-strict increases of S_i can only occur when moving from $S_{i,j}$ to $S_{i,j+1}$ for some j. Thus, $|S_i| < K+3X+n \leq K+3X+3m$.

Note that X, K and $|S_i|$ are all multiples of $3m$. For X, this is by definition, for K, it follows from the fact that each element of A is a multiple of $3m$, and for $|S_i|$ recall that each chunk C_j that contributes a nonempty subsequence of S_i contributes a sequence of length $X+a_j$. Therefore $|S_i| < K+3X+3m$ implies $|S_i| \leq K+3X$. Suppose there exists a partition cell S_j with $|S_j| < K+3X$, then $|S| < m(K+3X)$, which contradicts our assumption of $|S| = m(K+3X)$. Hence $|S_i| = K+3X$ as claimed. ◀

3 A Tight Upper Bound of the Circular Shifting Number

In this section, we investigate an upper bound of the circular shifting number and prove the following theorem.

▶ **Theorem 11.** *Given a drawing δ_G of an n-vertex outerplanar graph G, the circular shifting number* $\text{shift}^\circ(\delta_G) \leq n - \lfloor\sqrt{n-2}\rfloor - 2$*, and this bound is tight.*

To prove the upper bound, we present an untangling procedure that fixes at least $\lfloor\sqrt{n-2}\rfloor + 2$ vertices in the following. Given an n-vertex outerplanar graph $G=(V,E)$ and a circular drawing δ_G of G, we initially assume, without loss of generality, that in δ_G the vertices are in the clockwise order $v_1, v_2, \cdots, v_n$.

We first compute a crossing-free circular drawing δ_G^U of G in linear time [27]. The clockwise vertex ordering σ_G^U of δ_G^U is obtained by traversing vertices in δ_G^U clockwise starting from v_1. Now we consider untangling δ_G by moving vertices such that the vertices are ordered as σ_G^U clockwise or counterclockwise. Essentially, the problem of transforming the vertex order $v_1, v_2, \cdots, v_n$ to σ_G^U or its reversal with a minimum number of vertex moves is equivalent to finding a longest increasing subsequence of σ_G^U or its reversal, which can be fixed during the transformation.

It remains to prove that there is a strictly increasing subsequence of length $\lfloor\sqrt{n-2}\rfloor+2$ or a strictly decreasing subsequence of length $\lfloor\sqrt{n-2}\rfloor+2$ in the cyclic ordering σ_G^U, which follows from the Erdős–Szekeres theorem for cyclic permutations.

▶ **Theorem 12** (Erdős–Szekeres theorem for cyclic permutation [32]). *Given two integers s, r, any cyclic sequence of $n \geq sr+2$ different real numbers has an increasing cyclic subsequence of $s+2$ terms or a decreasing cyclic subsequence of $r+2$ terms, and this bound is tight.*

To show that this bound is tight, we construct an example drawing as follows. Let $x > 2$ be a positive integer and let G be the (x^2+1)-vertex cycle graph C_{x^2+1} that consists of the single cycle $v_0, \cdots v_{x^2}, v_0$ (in this order). Let σ_G be a cyclic permutation of numbers $0, 1, \cdots, x^2$ such that there is neither an increasing cyclic subsequence of $x+2$ terms nor a decreasing cyclic subsequence of $x+2$ terms. Such a permutation must exist since x^2+2 is a tight lower bound for the size of a cyclic permutation having such subsequences [32]. Let δ_G be a circular drawing of G where the vertices are placed as σ_G clockwise. We claim that $\text{shift}^\circ(\delta_G) \geq n - \lfloor\sqrt{n-2}\rfloor - 2$. In a crossing-free circular drawing of G, the vertices are ordered as $v_0, v_1, \cdots v_{x^2}$ clockwise or counterclockwise. Therefore, if we could untangle δ_G where K vertices remain fixed for a positive integer K, these K vertices must form a strictly increasing or decreasing subsequence in σ_G. Hence $K < x+2$ in σ_G.

4 A Tight Upper Bound for Almost-Planar Drawings

Let $G = (V, E)$ be an outerplanar graph, let δ_G be an almost-planar circular drawing of G. In this section, we present an untangling procedure for such almost-planar circular drawings that provides a tight upper bound of $\lfloor \frac{n}{2} \rfloor - 1$ on $\text{shift}^\circ(\delta_G)$.

▶ **Theorem 13.** *Given an almost-planar drawing δ_G of an n-vertex outerplanar graph G the circular shifting number $\text{shift}^\circ(\delta_G) \leq \lfloor \frac{n}{2} \rfloor - 1$, and this bound is tight.*

To see that the bound is tight, let $n \geq 4$ be an even number and let G be the cycle on vertices $v_1, \ldots, v_n, v_1$ (in this order) and let δ_G be a drawing with the clockwise order $v_2, \ldots, v_{2i} \ldots, v_n, v_{n-1}, \ldots, v_{2i+1}, \ldots, v_1$; see Figure 4.

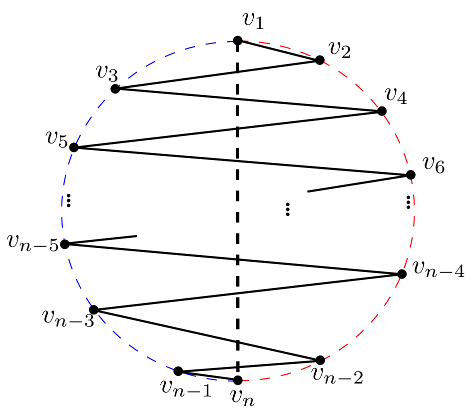
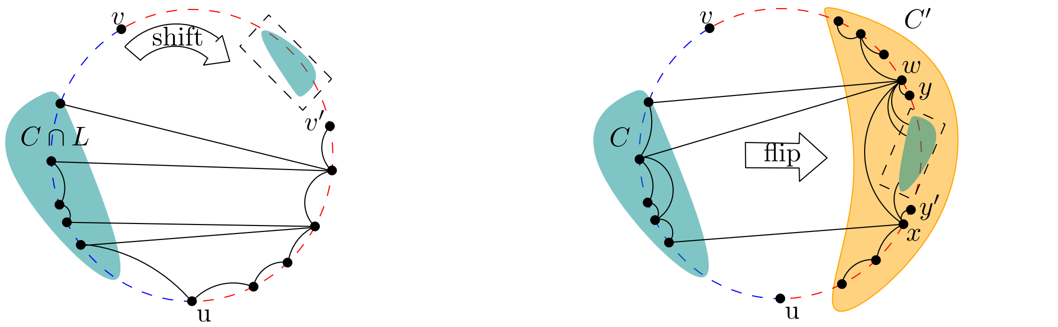


Figure 4 An almost-planar drawing δ_G with $\text{shift}^\circ(\delta_G) = \frac{n}{2} - 1$.

We claim that $\text{shift}^\circ(\delta_G) \geq \frac{n}{2} - 1$. Clearly, the clockwise circular ordering of its vertices in a crossing-free circle drawing is either $v_1, v_2, \ldots, v_n$ or its reversal. Assume that we turn it to the clockwise ordering $v_1, v_2, \ldots, v_n$; the other case is symmetric. In δ_G, the $\frac{n}{2}$ odd-index



(a) Case 1 **(b)** Case 2.2 (non-connecting component)

Figure 5 Moving a left component, keeping/reversing the clockwise ordering of its vertices.

vertices $v_1, \ldots, v_{2i+1} \ldots, v_{n-1}$ and v_n are ordered counterclockwise. To reach a clockwise ordering, we need to move all but two of these vertices. Thus, at least $\frac{n}{2} - 1$ vertices in total are required to move.

The remainder of this section is devoted to proving the upper bound. Let $e = uv$ be the edge of δ_G that contains all the crossings, and let $G' = G - e$ and $\delta_{G'}$ be the circular drawing of G' by removing the edge e from δ_G. The edge uv partitions the vertices in $V \setminus \{u, v\}$ into the sets L and R that lie on the left and right side of the edge uv (directed from u to v).

▶ **Theorem 14.** *Let δ_G be an almost-planar drawing of an outerplanar graph G. A planar circular drawing of G can be obtained by moving only vertices of L or only vertices of R to the other side in δ_G and fixing all the remaining vertices. The untangling moves only* $\min\{|L|, |R|\}$ *vertices and can be computed in linear time.*

This immediately implies the upper bound from Theorem 13, since $|L \cup R| = n - 2$, and therefore $\min\{|L|, |R|\} \leq \lfloor \frac{n}{2} \rfloor - 1$. To prove Theorem 14, we distinguish different cases based on the connectivity of u and v in G'.

Case 1: u, v are not connected in G'. Consider a connected component C of G' that contains vertices from L and from R.

▶ **Theorem 15.** *Suppose u, v are not connected in G'. Let C be a connected component of G' that contains vertices from L and from R. It is possible to obtain a new almost-planar drawing δ'_G of G from δ_G by moving only the vertices of $C \cap L$ (resp. $C \cap R$) such that C lies entirely on the right (resp. left) side of uv.*

Proof. Since u, v are not connected in G', C contains at most one of u, v. Without loss of generality, we assume that $v \notin C$; see Figure 5a. Let v' be the first clockwise vertex after v that lies in C. Let δ'_G be the drawing obtained from δ_G by moving the vertices of $C \cap L$ clockwise just before v' without changing their clockwise ordering. Observe that this removes all crossings of e with C. The choice of v' ensures that no edge of C alternates with an edge whose endpoints lie in $V \setminus C$. Finally, the vertices of C maintain their clockwise order. This shows that no new crossings are introduced, and the crossings between e and C are removed. ◀

By applying Proposition 15 for each connected component of G' that contains vertices from L and from R, we obtain a planar circular drawing of G.

Case 2: u, v are connected in G'. Let C be the connected component in G' that contains both vertices u and v. Note that if C' is another connected component of G', then it must lie entirely to the left or entirely to the right of edge e. Here, we ignore such components as they never need to be moved. We may hence assume that G' is connected.

Case 2.1: u, v are 2-connected in G'. We claim that in this case δ_G is already planar.

▶ **Theorem 16.** *If u and v are 2-connected in G', then δ_G is planar.*

Proof. If vertices $u, v \in V$ are 2-connected in G', then G' contains a cycle C that includes both u and v. In $\delta_{G'}$, this cycle is drawn as a closed curve. Any edge that intersects the interior region of this closed curve therefore has both endpoints on C. If there exists an edge $e' = xy$ that intersects $e = uv$, then contracting the four subpaths of C connecting each of $\{x, y\}$ to each of $\{u, v\}$ yields a K_4-minor in G, which contradicts the outerplanarity of G. ◀

Case 2.2: u, v are connected but not 2-connected in G'. In this case G' contains at least one cut-vertex that separates u and v. Notice that each path from u to v visits all such cut-vertices between u and v in the same order. Let f and l be the first and the last cut-vertex on any uv-path. Additionally, add u to the set of L, R that contains f and likewise add v to the set of L, R that contains l. Let X denote the set of edges of G' that have one endpoint in L and the other in R. Each connected component of $G' - X$ is either a subset of L or a subset of R, which are called *left* and *right components*, respectively. We call a component of $G' - X$ *connecting* if it contains either u or v, or removing it from G' disconnects u and v. For a left component C_L and a right component C_R, we denote by $E(C_L, C_R)$ the set of edges of G' that connect a vertex from C_L to a vertex in C_R. We can observe that since G' is connected, for any edge that connects a left and a right component, at least one of the components must be connecting. We use the following observation.

▶ **Observation 17.** *If P is an xy-path in a left (right) component C, then it contains all vertices of C that are adjacent to a vertex of a right (left) component and lie between x and y on the left (right) side.*

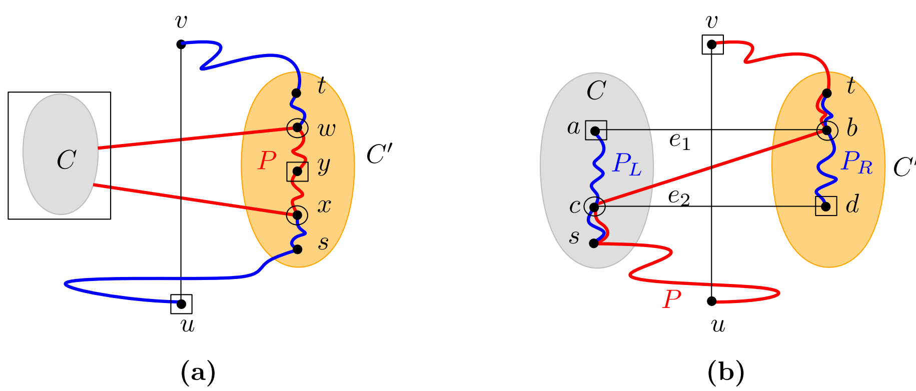


Figure 6 The $K_{2,3}$-minors we use in the proofs of (a) Lemma 18 and (b) Lemma 20.

▶ **Lemma 18.** *Every non-connecting component C of $G' - X$ is adjacent to exactly one component C' of $G' - X$. Moreover, C' is connecting, there are at most two vertices in C' that are incident to edges in $E(C, C')$, and if there are two such vertices $w, x \in C'$, then they are adjacent and removing wx disconnects C'.*

Proof. Without loss of generality, we assume that C is a left component. Since C is non-connecting, any component adjacent to it must be connecting. Moreover, if there are two distinct such components, they lie on the right side of the edge uv. Then either there is a path on the right side that connects them (but then they are not distinct), or removing C disconnects these components, and therefore uv, contradicting the assumption that C is a non-connecting component. Therefore C is adjacent to exactly one other component C', which must be a right connecting component. Let w and x be the first and the last vertex in C' that are adjacent to vertices in C when sweeping the vertices of G clockwise in δ_G starting at v; see Figure 6a. The lemma holds trivially if $w = x$. Suppose $w \neq x$. Next we show that $wx \in E$ and that wx is a bridge of C. Let P be an arbitrary path from w to x in C. If P contains an internal vertex y, then the path P together with a path from w to x whose internal vertices lie in C forms a cycle, where x and w are not consecutive. Note that at least one of u, v, say u, is not identical to w, x, otherwise, u, v are 2-connected. This cycle, together with disjoint paths from w to v and x to u and the edge uv yields a $K_{2,3}$-minor in G; see Figure 6a. Such paths exist, by the outerplanarity of $\delta_{G'}$ and the fact that C' is connecting, but C is not. Since G is outerplanar, and therefore cannot contain a $K_{2,3}$-minor, this immediately implies that P consists of the single edge wx, which must be a bridge of C' as otherwise there would be a wx-path with an internal vertex. Observation 17 implies that w and x are the only vertices of C that are adjacent to vertices in C'. ◀

▶ **Theorem 19.** *Let C be a left (right) non-connecting component of $G' - X$. It is always possible to obtain a new almost-planar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side.*

Proof. Without loss of generality, we assume that C is a left component. Since C is non-connecting, then by Lemma 18, it is adjacent to at most two vertices on the right side. If there are two such vertices, denote them by w and x such that w occurs before x on a clockwise traversal from v to u. Note that wx is a bridge of a right component C' by Lemma 18; see Figure 5b. Consider the two components of $C' - wx$ and let y be the last vertex that lies in the same component as w when traversing vertices clockwise from w to x. If C is connected to only one vertex, then we denote this by y. In both cases, let y' be the vertex of L that immediately succeeds y in clockwise direction (If $y = u$, let y' be the vertex that immediately precedes y.).

We obtain δ'_G by moving all vertices of $C \setminus \{u, v\}$ between y and y', reversing their clockwise ordering. Observe that the choice of y and y' guarantees that δ'_G is almost-planar and all crossings lie on uv. ◀

It remains to deal with connecting components.

▶ **Lemma 20.** *The connecting component of $G' - X$ containing u or v is adjacent to at most one connecting component. Every other connecting component is adjacent to exactly two connecting components. Moreover, if C and C' are two adjacent connecting components, then there is a vertex w that is incident to all edges in $E(C, C')$.*

Proof. The claims concerning the adjacencies of the connecting components follows from the fact that every uv-path visits all connecting components in the same order. It remains to prove that all edges between two connecting components share a single vertex. If u and v are in one component, then this component is the only connecting component and there is nothing to show.

Now let C and C' be adjacent connecting components and assume that C or C' may contain one of u or v but not both. Furthermore, we assume without loss of generality, that

C is a left and C' is a right component. For the sake of contradiction, assume there exist two edges $e_1, e_2 \in E(C, C')$ that do not share an endpoint. Let $e_1 = ab$ and $e_2 = cd$ where $a, c \in C$ and $b, d \in C'$ such that their clockwise order is a, b, d, c; see Figure 6b. Note that one of u, v is not in the set $\{a, b, c, d\}$. Otherwise, u and v are 2-connected, which contradicts our case assumption. In the following, we assume without loss of generality that a, b, c, d, v are five distinct vertices in G'. Let P be a path from u to v in G'. Since C and C' are both connecting, P contains vertices from both components. When traversing P from u to v, let s and t denote the first and the last vertex of $C \cup C'$ that is encountered, respectively. Here, we assume without loss of generality that $s \in C$ and $t \in C'$. Let P_L be a path in C that connects s to a and let P_R be a path in C' that connects d to t. By Observation 17, P_L contains c and P_R contains b. We then obtain a $K_{2,3}$-minor of G by contracting each of the paths $P_L[c, a]$, $P_R[d, b]$, $vuP[u, s]P_L[s, c]$, and $P_R[b, t]P[t, v]$ into a single edge. ◀

By Lemma 18 and Lemma 20, all vertices of a connecting component of $G' - X$ can be moved to the other side, similarly as in Theorem 19.

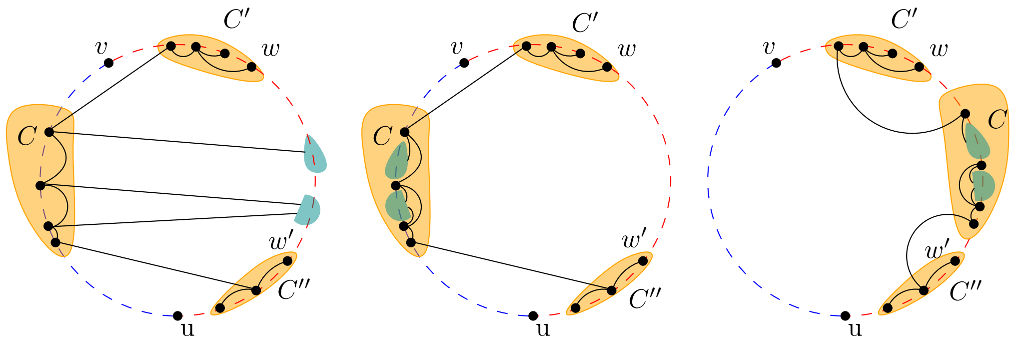
▶ **Theorem 21.** *Let C be a left (right) connecting component of $G' - X$. It is possible to obtain a new almost-planar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side.*

Proof. We assume w.l.o.g. that C is a left connecting component. Now, we determine two vertices w and w' such that a right component is a non-connecting component adjacent to C iff it lies between w and w' entirely. If u, v are not in C, by Lemma 20, C is adjacent to exactly two right connecting components C', C''(see Figure 7b). In the following, we assume that v, C', C'', u are in clockwise order. Let w be the last vertex in C' and w' be the first vertex in C'' when traversing the vertices in δ_G clockwise from v; If C contains both u and v, let w be v and w' be u; If C contains either u or v, by Lemma 20, C is adjacent to exactly one right connecting components C'. Assume w.l.o.g. that $v \in C$. Let w be the last vertex in C' when traversing the vertices in δ_G clockwise and w' be u. Observe that, due to the connectivity of G' and the outerplanarity of $\delta_{G'}$, each right component that entirely lies between w and w' is a non-connecting component adjacent to C. Again, we want to only move the component C to the right side between w and w' without introducing any crossings. For simplicity, we describe the procedure in two phases. In the first phase, we move all the right non-connecting components connected to C to the left side "temporarily" as the procedure described in the proof of Lemma 19 such that the components are merged into C on the left; see Figure 7c. In the second phase, we move the component $C \setminus \{u, v\}$ (alongside the vertices that are moved in the first phase) to the right side between w and w', reversing their clockwise ordering; see Figure 7d. For each right component C' that is adjacent to C, by Lemma 20, there is exactly one vertex shared by edges $E(C, C')$. Thus, there is no crossing on the right side of uv after the second phase. Furthermore, the vertices moved to the left at the first phase are in the same order as in δ_G after two reversals and they still lie between w and w'. Therefore, we could reach the same order after this two-phase procedure by only moving the vertices in C to the right side accordingly. ◀

Theorem 19 and Theorem 21 together imply Theorem 14.

5 Untangling Almost-Planar Drawings

In this section, we consider how to untangle an almost-planar circular drawing δ_G of an n-vertex outerplanar graph $G = (V, E)$ with the minimum number of vertex moves. Firstly,



(a) (b) 1.Phase (c) 2.Phase

Figure 7 Case 2.2 (connecting-component) **(a)** A left connecting component C that is adjacent to vertices on the right side. **(b)** Moving all the right non-connecting components connected to C to the left side "temporarily" in the 1.Phase. **(c)** Moving the component C (alongside the vertices that are moved in the 1.Phase) to the right side, reversing their clockwise ordering.

we study this problem in several restricted settings (Sections 5.1–5.3), which leads us to the design of an $O(n^2)$-time algorithm to compute $\text{shift}^\circ(\delta_G)$ in Section 5.4. Let $e = uv$ be the edge of δ_G that contains all the crossings, and let $G' = G - e$ and $\delta_{G'}$ be the straight-line circular drawing of G' by removing the edge e from δ_G. The edge uv partitions the vertices in $V \setminus \{u, v\}$ into the sets L and R that lie on the left and right side of the edge uv (directed from u to v). Let C_u and C_v be the connected components of G' that contain u and v, respectively. Note that $C_u = C_v$ if u, v are connected.

5.1 Fixed Edge Untangling

Here we consider untangling under the restriction that the positions of u and v are fixed. We denote such untangling as *fixed edge untangling*. From very similar arguments as in Section 4, we derive the following statements.

▶ **Observation 22.** *Let C be a connected component of G'. Every fixed edge untangling procedure musts either move all vertices in L_C (R_C) to the right (left) side.*

Equipped with Theorem 19 and Theorem 21 in Section 4, we get the following Lemma 23.

▶ **Lemma 23.** *Let C be a connected component of G'. It is always possible to obtain an almost-planar drawing δ'_G of G from δ_G by moving all vertices in $L \cap C$ (resp. $R \cap C$) to the right (resp. left) side.*

Proof. If u, v are not connected in G', the claim is exact the same as Theorem 15. We now consider the case that u, v are connected in G'. Let C be the connected component of G' that contains both u and v. We could always move either L_C or R_c by Theorem 19 and Theorem 21. Note that any other connected component C' of G' must lie entirely to the left or entirely to the right of edge e since $\delta_{G'}$ is planar and u, v are connected in G'. ◀

▶ **Theorem 24.** *Given an almost-planar drawing δ_G of an outerplanar graph G, a fixed edge untangling of δ_G with the minimum number of vertex moves can be computed in linear time.*

Proof. According to the Observation 22, it is necessary to move either the entire set L_C or the entire set R_C in each component C of G'. Thus, a fixed edge untangling with the minimum number of moves always move the smaller one between L_C and R_c in each component C. By Lemma 23, applying corresponding untangling procedures described in the proofs of Theorem 15, Theorem 19 and Theorem 21, we could compute a sequence of moves with the minimum number of vertex moves and report them in linear time. ◀

5.2 Single Component Untangling

Next, we study an untangling variant, called *Single Component Untangling*, which moves vertices of one particular connected component of G' that contains the vertices u or v, while the other components remain fixed. We claim that δ_G can always be untangled in this way.

▶ **Theorem 25.** *It is always possible to untangle δ_G by moving only the vertices of C_u or only the vertices of C_v and such a single component untangling procedure can be found in linear time.*

Proof. If $C_u = C_v$ the claim is trivially true. So let's consider the case that u and v are not connected in G' and assume that $|C_u| \leq |C_v|$. We move the vertices of C_u as follows. Let σ_u be the clockwise order of C_u in $\delta_{G'}$, starting with u. We insert the vertices of C_u in the order σ_u clockwise right after v to obtain a new drawing $\delta'_{G'}$ of G'. Since C_u was crossing-free before and is placed consecutively on the circle, it remains crossing-free. No other edges have been moved. Furthermore, u and v are now neighbors on the circle, so we can insert the edge uv without crossings and have untangled δ_G with $\min\{|C_u|, |C_v|\}$ moves. ◀

5.3 Component-Fixed Untangling

An untangling under the restriction that both of C_u and C_v must contain fixed vertices, is denoted as *Component-Fixed Untangling*.

We introduce some notions and provide basic observations. Let G be a connected outerplanar graph. Let B be a 2-connected component of G and $E(B)$ the set of edges in B. Since G is connected and B is 2-connected, each connected component of $G - E(B)$ contains exactly one vertex in B. Given a vertex b in B, let C_b be the connected component of $G - E(B)$ that contains b. We denote C_b as the *attachment* of the 2-connected component B at the vertex b.

Let $H(B)$ be the cyclic vertex ordering of B in the order of its Hamiltonian cycle[1]. We get Observation 26; see Figure 8.

▶ **Observation 26.** *Let δ_G be a planar circular drawing of an outerplanar graph G and B be a 2-connected component of G. Then, the clockwise cyclic vertex ordering of B in δ_G is either $H(B)$ or its reverse. Furthermore, for each attachment of B, its vertices appear consecutively on the circle in δ_G.*

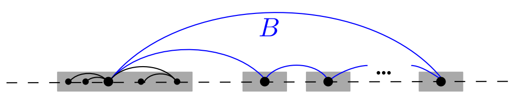


Figure 8 A 2-connected component B (in blue) and its attachments (gray boxes) in a planar circular drawing.

[1] In every outerplanar biconnected graph, there is a unique Hamiltonian cycle that visits each node exactly once [28].

Given a connected outerplanar graph G, a 2-connected component B of G and a circular drawing δ_G, we say a sequence S of vertex moves of G is *canonical*, associated with B, if in the drawing obtained by applying S to δ_G, the clockwise cyclic vertex ordering of each attachment of B remains unchanged. Now we are ready to show that an optimal component-fixed untangling with the restriction that fixed vertices exist in both of C_u and C_v can be found in $O(n^2)$ time; see Theorem 27.

▶ **Theorem 27.** *A component-fixed untangling procedure U with the minimum number of vertex moves can be found in $O(n^2)$ time.*

The reminder of this section is devoted to describing the procedure U. We distinguish between the following two cases based on the connectivity of u, v in G'. In each case, we present a procedure that runs in $O(n^2)$ time and reports an optimal component-fixed untangling procedure.

Case 1: u and v are connected in G'. Let C be a connected component of G' that does not contain u, v. We claim now that C must lie entirely on one side of uv in δ_G. Otherwise, let P be a path of $\delta_{G'}$ that connects u and v. Then there would exist crossings between edges of P and edges of C in $\delta_{G'}$ which contradicts the fact that $\delta_{G'}$ has no crossings. Thus, we can ignore such components as they do not need to be involved in an untangling. Hence, we may assume G' is a connected graph. If u and v are 2-connected in G', then δ_G is already outerplanar; see Theorem 16. Now we consider the case that u and v are connected, but not 2-connected in G'. Note that u, v are 2-connected in G. Let B be the 2-connected component of G that contains u, v. We prove that each component-fixed untangling U can be transformed into a canonical untangling with smaller or the same number of vertex moves; see Lemma 28. Thus, we restrict our attention to canonical untanglings. Let $H(B) = b_1, \dots b_k$ be the cyclic vertex ordering of the Hamiltonian cycle of B. Let A_i be the attachment of B at the vertex b_i and let $\sigma(A_i)$ be the clockwise vertex ordering of A_i in δ_G for $i \in \{1, \dots, k\}$. We consider an optimal canonical component-fixed untangling U_o which orders B clockwise as $H(B)$. Let $\delta_G^{''}$ be the outerplanar drawing obtained by applying U_o. Then the clockwise vertex ordering of $\delta_G^{''}$ is exactly the concatenation of $\sigma(A_1), \sigma(A_2), \dots, \sigma(A_k)$. Given $\delta_G^{''}$, an optimal untangling transforming δ_G to $\delta_G^{''}$ can be computed in $O(n^2)$ time; see [23]. Analogously, we obtain an optimal component-fixed untangling U_r which orders B counterclockwise as $H(B)$. From the two untanglings U_o and U_r, we report the one which moves less vertices as the optimal component-fixed untangling.

▶ **Lemma 28.** *Let B be the 2-connected component of G that contains u, v. Each component-fixed untangling U of δ_G can be transformed into a canonical vertex move sequence U_c (associated with B) that untangles δ_G. Furthermore, the number of vertex moves in U_c is not greater than the number of vertex moves in U.*

Proof. Given a component-fixed untangling U of δ_G, let δ_G^U be the drawing obtained after applying U on δ_G. In δ_G^U, the cyclic vertex ordering of B (clockwise or counterclockwise) must correspond to its Hamiltonian cycle ordering $H(B)$. Furthermore, the vertices of each attachment of B appear consecutively in δ_G^U, including one vertex of B; see Observation 26. Let $A_1, \dots, A_k$ be the attachments of B in G (indexed in clockwise order as in δ_G^U) and let $\sigma(A_i)$ be the clockwise vertex ordering of A_i in δ_G for $i \in \{1 \dots k\}$. Now consider the vertex ordering $\sigma'_G = (\sigma(A_1), \cdots, \sigma(A_k))$ and let δ'_G be an arbitrary circular drawing where the vertices are ordered as σ'_G. Note that the vertex ordering of each attachment is $\sigma(A_i)$ in δ'_G as in the almost-planar drawing δ_G, thus each attachment in δ'_G is crossing-free. Moreover, in δ'_G the vertices of B are ordered as in the planar drawing δ_G^U, thus there is no crossing inside

B. Overall, δ'_G is a planar circular drawing. Let U_c be the untangling of δ_G with minimum number of vertex moves such that the clockwise vertex ordering of the resulting drawing is σ'_G.

To see that U_c does not move more vertices than U, let σ_G and σ_G^U be the clockwise vertex orderings of δ_G and δ_G^U, respectively. We can observe that any common subsequence of σ_G, σ_G^U is a subsequence of σ'_G. ◀

Case 2: u and v are not connected in G'. Note that a connected component of G' that lies entirely on one side of uv in δ_G can be ignored, since there is no need to move any vertices in such components. After ignoring such components, we can assume that a connected component C of G' either contains u, v or C contains vertices from L and also vertices from R.

▶ **Observation 29.** *In $\delta_{G'}$, vertices of C_u (resp. C_v) lie consecutively on the cycle.*

The first step of our untangling procedure U deals with the connected components of G' that neither contain u nor v. Let U^{fix} be an arbitrary component-fixed untangling of δ_G, and let δ_G^{fix} be the outerplanar drawing of G obtained from δ_G by applying U^{fix}.

▶ **Lemma 30.** *Let C be a connected component of G' that does not contain vertices u or v. Let f_u, f_v be two vertices in C_u and C_v, respectively, which are fixed in δ_G^{fix}. Then, C must lie entirely on one side of $f_u f_v$[2] in δ_G^{fix}.*

Proof. In the graph G, due to the definition of f_u and f_v, there exists a path P_1 in C_u connecting f_u to u, and a path P_2 in C_v connecting v to f_v; see Figure 9. Then, the path $P = P_1 uv P_2$ in G connects f_u to f_v. In δ_G^{fix}, suppose that the connected component C is not entirely on one side of $f_u f_v$, it implies that at least one edge xy in C has endpoints x, y alternate with f_u, f_v in clockwise ordering of δ_G^{fix} and then has crossings with P. It contradicts the outerplanarity of the drawing δ_G^{fix}. ◀

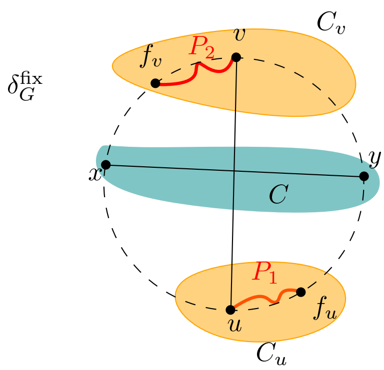


Figure 9 An example illustration for the proof of Lemma 30.

Now let C be a connected component that does not contain $u.v$. Vertices f_u and f_v partition the vertices of C in drawing δ_G into two sets L_C and R_C that are encountered clockwise and counter-clockwise from f_u to f_v in δ_G, respectively. Observe that, $L_C = L \cap C$

[2] Given a circular drawing of $G = (V, E)$, two vertices a, b partitions the vertices in $V \setminus \{a, b\}$ into two sets that lie on the left side and right side of the ray $\overrightarrow{ab}$.

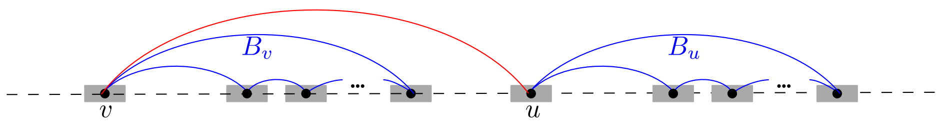


Figure 10 In any clockwise vertex ordering of a planar circular drawing, u, v must be the extreme vertices in the 2-connected components B_v and B_u, respectively

and $R_C = R \cap C$; see Observation 29. Let $m(C) = \min\{|L \cap C|, |R \cap C|\}$. By Lemma 30, $m(C)$ is a lower bound of the moved vertices in C in a component-fixed untangling. By Lemma 23, there is a procedure moving $m(C)$ vertices of C such that C lies entirely on one side of uv. In the first step of our untangling procedure U, we repeat this step for each component not containing u or v. After that, an almost-planar drawing of G remains that has already each component not containing u, v placed entirely on one side of uv. We can ignore such components from now on since they never need to be moved again.

Now we assume that G' has exactly two connected components, namely C_u and C_v. Consider an arbitrary outerplanar drawing δ'_G of G. Let $\sigma(\delta'_G)$ be the circular ordering of vertices in δ'_G encountered clockwise. Observe that, in $\sigma(\delta'_G)$, the vertices of C_u (resp. C_v) are in a consecutive subsequence $\sigma(C_u)$ (resp. $\sigma(C_v)$). Otherwise, alternating vertices of two connected components would introduce crossings.

Given an edge e' in C_v, we say e' *covers* v if the endpoints of e alternate with u and v in $\delta_{G'}$. Note that there is no edge covering v in $\sigma(C_v)$. Otherwise, such an edge would cross with edge uv. Therefore, in a valid untangling of δ_G, it is necessary to move vertices of C_v in δ_G such that no crossing is introduced in C_v and v is not covered by any edges in C_v. Similarly, the same claim holds also for C_u. We call such vertex moves *vertex unwrapping*. In the following, we consider how to find a valid unwrapping of v with the minimum number of vertex moves. The same operation will be also applied to C_u. Observe that, once u, v are both unwrapped, adding the edge e into the drawing does not introduce any crossings. The combination of these two unwrappings makes an optimal untangling. Here, we also consider the canonical vertex sequences and get the following Lemma 32. The proof is quite similar to the proof of Lemma 28 which concerns canonical untanglings.

▶ **Observation 31.** *There exists at least one 2-connected component B of C_v such that B contains v and no edge in the attachment of v (associated with B) covers v in δ_G.*

The reason for this observation is that either no 2-connected component B containing v contains an edge covering v, in which case v is already unwrapped and the statement is true for any such B. Or some 2-connected component B does contain a covering edge, but then the attachment of v in B cannot cover v due to planarity of $\delta_{G'}$.

▶ **Lemma 32.** *Let B be a 2-connected component of C_v that contains v such that the attachment of v contains no edge covering v. Each unwrapping U of v can be transformed into a canonical unwrapping U_c (associated with B). Furthermore, the number of vertex moves in U_c is not greater than the number of vertex moves in the original unwrapping U.*

Proof. Given a unwrapping procedure U of v, let δ^U_G be the drawing obtained after applying U on δ_G. In δ^U_G, the cyclic vertex ordering of B (clockwise or counterclockwise) must correspond to its Hamiltonian cycle ordering $H(B)$. Furthermore, the vertices of each attachment of B appear consecutively in δ^U_G, including one vertex of B; see Observation 26. Let $A_1, ...A_k$ be the attachments of B in C_v (in this clockwise order in δ^U_G), let $\sigma(A_i)$ be the clockwise vertex ordering of A_i in δ_G for $i \in \{1 \dots k\}$. Consider the clockwise vertex ordering σ'_G

where the vertices of $B \cup C_u$ are ordered as in δ^U_G. Furthermore, for each attachment A_i the vertices of A_i appear consecutively in the clockwise ordering $\sigma(A_i)$. Let δ'_G be an arbitrary circular drawing where the vertices are ordered as σ'_G. Note that the vertex ordering of each attachment of B is $\sigma(A_i)$ in δ'_G as in the almost-planar drawing δ_G, thus each attachment in δ'_G is crossing-free. Moreover, in δ'_G the vertices of B are ordered as in the planar drawing δ^U_G, thus there is no crossing inside B. Overall, the vertex v is unwrapped in δ'_G. It remains to prove that the untangling U', which transforms δ_G to δ'_G, moves less than or equally many vertices as U. By construction each common subsequence of δ_G and δ^U_G is also a subsequence of δ'_G, which implies this fact. ◀

By Lemma 32, we restrict our attention to canonical unwrappings. Fixing a 2-connected component B_v of C_v containing v such that no edge in the attachment (associated with B_v) of v covers v, we consider the two possible canonical unwrappings of v, which respectively order vertices of B clockwise along $H(B)$ or its reversal, and compute the corresponding resulting clockwise vertex ordering σ_v and σ^{rev}_v of C_v. With the same idea, we get the clockwise vertex orderings σ_u and σ^{rev}_u of C_u by the canonical unwrappings of u. We then get the four optimal unwrappings, each of them transforming δ_G to one of the vertex orderings $(\sigma_v\sigma_u), (\sigma^{rev}_v\sigma_u), (\sigma_v\sigma^{rev}_u)$ and $(\sigma^{rev}_v\sigma^{rev}_u)$. Such optimal unwrappings can be computed in $O(n^2)$ time; see [23]. We report the one that moves the minimum number of vertices as an optimal component-fixed untangling.

5.4 Circular Untangling

Given an almost-planar drawing δ_G, we claim that it is always possible to compute an optimal untangling procedure for δ_G in $O(n^2)$ time, where n is the number of vertices of G. In our approach, we use procedures described in Sections 5.1–5.3 as subroutines.

The Approach. *Step 1*: we compute an optimal component-fixed untangling U by applying the approach described in Section 5.3. An optimal component-fixed untangling U can be reported in $O(n^2)$ time (see Theorem 27).*Step 2*: let $m(U)$ be the number of vertex moves in U. we compare $m(U)$ with $\min\{|C_u|, |C_v|\}$. If $m(U) \leq \min\{|C_u|, |C_v|\}$, then we report U. Otherwise, if $m(U) > \min\{|C_u|, |C_v|\}$, we know U is not an optimal untangling procedure. Because there exists a specific untangling procedure U' which moves exactly $\min\{|C_u|, |C_v|\}$ vertices; see its description in the proof of Theorem 25. In this case, we compute and report this procedure U'. The second step takes linear time. In total, the whole procedure needs $O(n^2)$ time.

Correctness. Let U_a be the untangling reported by our approach. Now, we show that U_a is indeed an optimal untangling of δ_G by contradiction. Note that U_a has size bounded by $\min\{|C_u|, |C_v|\}$ (*Step 2*). Suppose there exists an untangling $U_{a'}$ which moves less vertices than U_a. Then $U_{a'}$ moves less vertices than $\min\{|C_u|, |C_v|\}$. If so, there are vertices in both of $|C_u|, |C_v|$ that remain fixed in $U_{a'}$. Thus, $U_{a'}$ is a component-fixed untangling. It leads to a contradiction to the fact that U_a has its size bounded by the size of optimal component-fixed untangling (*Step 1*). Therefore, U_a is indeed an untangling of δ_G with the minimum number of vertex moves.

▶ **Theorem 33.** *Given an almost-planar drawing δ_G of an outerplanar graph G, an untangling of δ_G with the minimum number of vertex moves can be computed in $O(n^2)$ time, where n denotes the number of vertices in G.*

6 Conclusions and Discussions

We introduced and investigated the problem of untangling non-planar circular drawings. First from the computational side, we demonstrated the NP-hardness of the problem Circular Untangling. Second, we studied the almost-planar circular drawings, where all crossings involve a single edge. We gave a tight upper bound of $\lfloor \frac{n}{2} \rfloor - 1$ on the shift number and an $O(n^2)$-time algorithm to compute it. Open problems for future work include: (i) The parameterized complexity of computing the circular shifting, e.g., with respect to the number of crossings or the number of connected components. (ii) Generalization of our results for almost-planar drawings. (iii) Investigation of minimum untangling by other elementary moves such as swapping vertex pairs or moving larger chunks of vertices.

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