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Untangling Almost Outerplanar Drawings

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Abstract

Given an n-vertex outerplanar graph G, let δ_G be a straight-line drawing of G, where the vertices lie on a circle and all crossings involve a single edge. We call such a drawing an *almost outerplanar* drawing. An outerplanar drawing of G can be obtained from δ_G by *untangling* it, i.e., moving the vertices on the circle in δ_G. Let $\mathrm{fix}^\circ(\delta_G)$ denote the maximum number of vertices that can remain fixed to untangle δ_G. We show $\mathrm{fix}^\circ(\delta_G) \geq \lceil (n+2)/2 \rceil$ and this bound is asymptotically tight.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Graph drawing, straight-line drawing, planarity, moving vertices, untangling

Acknowledgements Research supported by the Austrian Science Fund (FWF), grant P 31119

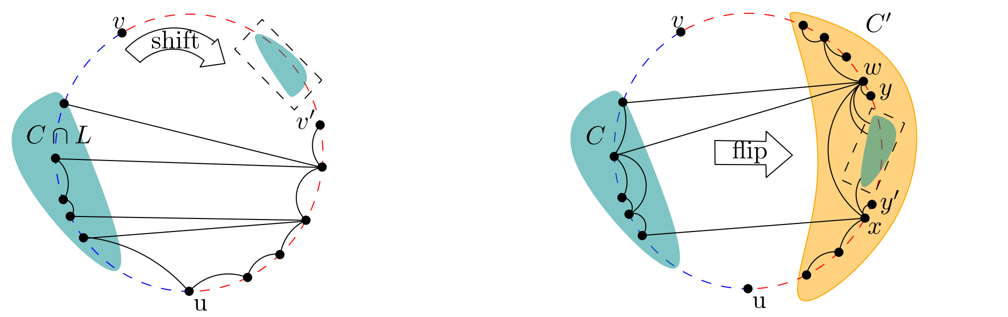
1 Introduction

A graph is an *outerplanar* graph if it has a planar drawing in which all vertices are on the boundary of a single face, and such a drawing is known as an outerplanar drawing. Given an n-vertex outerplanar graph G, let δ_G be a straight-line drawing of G, where the vertices lie on a circle and all crossings involve a single edge. We call δ_G an *almost outerplanar* drawing. Since G is outerplanar, an outerplanar drawing can be obtained from δ_G by moving the vertices on the circle. We call such a sequence of vertex moving operations an *untangling* of δ_G. We define the *outerplanar fixing number* $\mathrm{fix}^\circ(\delta_G)$ of an almost outerplanar drawing δ_G to be the maximum number of vertices that can remain fixed in an untangling of δ_G. The notion of untangling is often used in the literature for a crossing elimination procedure that makes a non-planar drawing of a planar graph crossing-free; see [**?**, 6, 7, 10, 18, 24, 25, 29]. Here, we follow an untangling procedure to obtain an outerplanar drawing from an almost outerplanar drawing.

2 Lower Bound for $\mathrm{fix}^\circ(\delta_G)$

In the following let $G = (V, E)$ be an outerplanar graph, let δ_G be an almost outerplanar drawing of G, let $e = uv \in E$ be the edge that contains all the crossings in δ_G, and let $G' = G - e$ and $\delta_{G'} = \delta_G - e$. The edge e partitions the vertices in $V \setminus \{u, v\}$ into the sets L and R that lie left and right of the edge uv (in the direction from u to v). We claim that it is possible to move vertices of L to the right side without modifying the order of $R \cup \{u, v\}$ to obtain an outerplanar drawing. By symmetry, it is also possible to just move vertices of R to the left side. The claimed bound then follows from the fact that $\min\{|L|, |R|\} \leq \lfloor (n-2)/2 \rfloor$. We distinguish cases based on the connectivity of u and v in G'.

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Access; Article No. 210; pp. 210:1–210:8
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



(a) Case 1 **(b)** Case 2.2 (non-connecting component)

Figure 1 Moving a left component, keeping/reversing the clockwise ordering of its vertices.

Case 1: u, v are not connected. Consider a connected component C of G' that contains vertices from L and from R. In this case, C contains at most one of u, v. W.l.o.g., assume $v \notin C$; see Figure 1a. Let v' be the first clockwise vertex after v that lies in C. Let δ'_G be the drawing obtained from δ_G by moving the vertices of $C \cap L$ clockwise just before v' without changing their clockwise ordering. Observe that this removes all crossings of e with C.

Case 2: u, v are connected. Now assume there exists a connected component in G' that contains both u and v. Note that if C' is a different connected component of G', then it must lie entirely to the left or entirely to the right of e. We ignore such components as they never need to be moved. We hence assume that G' is connected.

Case 2.1: u, v are 2-connected. Then δ_G is already planar; see Lemma 6 in Appendix A.

Case 2.2: u, v are connected but not 2-connected. G' contains at least one cutvertex that separates u and v. Notice here, each path from u to v visits all these cutvertices between u and v in the same order. Let f and l be the first and the last cutvertex on any uv-path, respectively. Additionally, add u to the set of L, R that contains f and likewise add v to the set of L, R that contains l. Let X denote the set of edges of G' that have one endpoint in L and the other in R. Each connected component of $G' - X$ is either a subset of L or a subset of R. we call these *left* and *right components*, respectively. We call a component of $G' - X$ *connecting* if it either contains u or v, or removing it from G' disconnects u and v. For a left component C_L and a right component C_R, we denote by $E(C_L, C_R)$ the edges of G' that connect a vertex from C_L to a vertex in C_R. We refer for the proofs to Appendix A.

▶ **Lemma 1.** *Every non-connecting component C is adjacent to exactly one component C' of $G' - X$. Moreover, C' is connecting, there are at most two vertices in C' that are incident to edges in $E(C, C')$, and if there are two such vertices $w, x \in C'$, then they are adjacent and removing wx disconnects C'.*

▶ **Theorem 2.** *Let C be a left (right) non-connecting component. It is always possible to obtain a new almost outerplanar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side.*

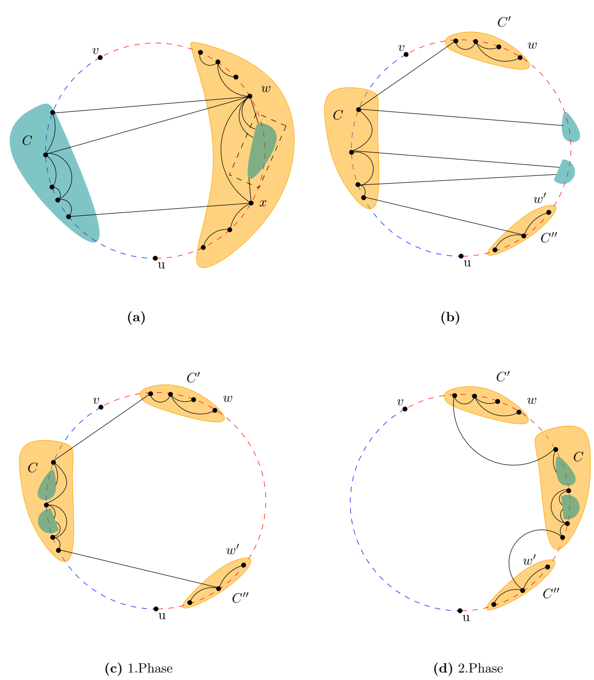
Proof. If C is non-connecting, then by Lemma 1, it is adjacent to at most two vertices in C' that are adjacent to C. If there are two such vertices, denote them by w and x. Note that w and x are consecutive in the drawing δ_G, since G' is connected and wx is a bridge by Lemma 1. Otherwise let w be the only such vertex and let x be a vertex on the right side that immediately precedes or succeeds x; see Figure 1b. We obtain δ'_G by moving all vertices of $C \setminus \{u, v\}$ between x and w, reversing their clockwise ordering. Observe that the choice of w and x guarantees that δ'_G is almost outerplanar and all crossings lie on uv. ◀

▶ **Lemma 3.** *The connecting component containing u or v is adjacent to at most one connecting component. Every other connecting component is adjacent to exactly two connecting components. Moreover, if C and C' are two adjacent connecting components, then there is a vertex w that is shared by all edges in $E(C, C')$.*

▶ **Theorem 4.** *Let C be a left (right) connecting component. It is always possible to obtain a new almost outerplanar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side.*

3 The Lower Bound is Tight

Let $n \geq 4$ be an even number and let G be the cycle on vertices $v_1, \ldots, v_n, v_1$ (in this order) and let δ_G be a drawing with the clockwise order $v_2, \ldots, v_{2i} \ldots, v_n, v_{n-1}, \ldots, v_{2i+1}, \ldots, v_1$; see Figure 2. Clearly, the clockwise circular ordering of its vertices in a crossing-free circle drawing is either $v_1, v_2, \ldots, v_n$ or its reversal. Assume that we turn it to the clockwise ordering $v_1, v_2, \ldots, v_n$; the other case is symmetric. In δ_G, the $\frac{n}{2}$ odd-index vertices $v_1, \ldots, v_{2i+1} \ldots, v_{n-1}$ and v_n are ordered counterclockwise. To reach a clockwise ordering, at most two of these vertices can be fixed. Thus, at most $n/2 + 1$ vertices in total can be fixed.



(a) **(b)**

(c) 1.Phase **(d)** 2.Phase

Figure 2 The drawing δ_G of the graph G defined in Section 3. It shows that $\text{fix}^\circ(\delta_G) \leq \frac{n+2}{2}$.

Open problems for future work. (i) The complexity of computing the outerplanar fixing number. (ii) Generalization of our result to non-outerplanar drawings of outerplanar graphs.

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A Omitted Proofs and Details

▶ **Observation 5.** *If P is an xy-path in a left (right) component C, then it contains all vertices of C that are adjacent to a vertex of a right (left) component and lie between x and y on the left (right) side.*

▶ **Lemma 6.** *If u and v are 2-connected in G', then δ_G is planar.*

Proof. If vertices $u, v \in V$ are 2-connected in G', then G' contains a cycle C that includes both u and v. In $\delta_{G'}$, this cycle is drawn as a closed curve. Any edge that intersects the interior region of this closed curve therefore has both endpoints on C. If there exists an edge $e' = xy$ that intersects $e = uv$, then contracting the four subpaths of C connecting each of $\{x, y\}$ to each of $\{u, v\}$ yields a K_4-minor in G, which shows that G is not outerplanar. ◀

▶ **Lemma 1.** *Every non-connecting component C is adjacent to exactly one component C' of $G' - X$. Moreover, C' is connecting, there are at most two vertices in C' that are incident to edges in $E(C, C')$, and if there are two such vertices $w, x \in C'$, then they are adjacent and removing wx disconnects C'.*

Proof. W.l.o.g., we assume that C is a left component. Since C is non-connecting, any component adjacent to it must be connecting. Moreover, if there are two distinct such components, they lie on the right side of the edge uv. Then either there is a path on the right side that connects them (but then they are not distinct), or removing C disconnects these components, and therefore uv, contradicting the assumption that C is a non-connecting component. Therefore C is adjacent to exactly one other component C', which must be a right connecting component. Let w and x be the first and the last vertex in C' that are incident to vertices in C when sweeping the vertices of G clockwise in δ_G starting at v; see Figure 3a. The lemma holds trivially if $w = x$. Suppose $w \neq x$. In the following we show that $wx \in E$ and that wx is a bridge of C.

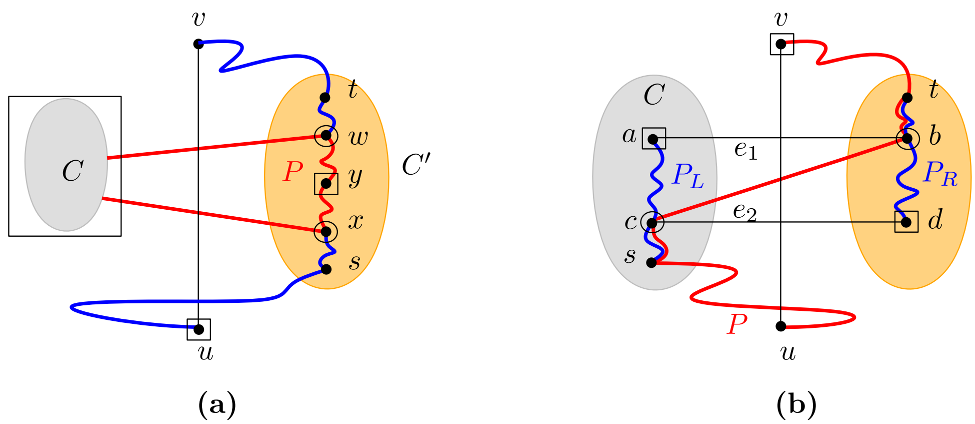
Let P be an arbitrary path from w to x in C. If P contains an internal vertex y, then the path P together with a path from w to x whose internal vertices lie in C forms a cycle where x and w are not consecutive. Note that at least one of u, v, say u, is not identical to w, x, otherwise, u, v are biconnected. This, together with disjoint paths from w to v and x to u and the edge uv yields a $K_{2,3}$-minor in G. Such paths exist, by the outerplanarity of $\delta_{G'}$ and the fact that C' is connecting, but C is not.

Since G is outerplanar, and therefore cannot contain a $K_{2,3}$-minor, this immediately implies that P consists of the single edge wx, which must be a bridge. Observation 5 implies that w and x are the only vertices of C that are adjacent to vertices in C'. ◀

▶ **Lemma 3.** *The connecting component containing u or v is adjacent to at most one connecting component. Every other connecting component is adjacent to exactly two connecting components. Moreover, if C and C' are two adjacent connecting components, then there is a vertex w that is shared by all edges in $E(C, C')$.*

Proof. The claims concerning the adjacencies of the connecting components follows from the fact that every uv-path visits all connecting components in the same order. It remains to prove that all edges between two connecting components share a single vertex.

Let C and C' be adjacent connecting components. If u and v are in one component, then this component is the only connecting component, and the claim holds vacuously. Now we assume that C or C' may contain u or v but not both. Furthermore, we may assume w.l.o.g., that C is a left and C' is a right component.



186 **Figure 3** The $K_{2,3}$-minors we use in the proofs of lemmas in Section 2.

Assume for the sake of contradiction that there exist two edges $e_1, e_2 \in E(C, C')$ that do not share an endpoint. Let $e_1 = ab$ and $e_2 = cd$ where $a, c \in C$ and $b, d \in C'$ such that their clockwise order is a, b, d, c; see Figure 3b. Note that one of u, v is not in the set $\{a, b, c, d\}$. Otherwise, u and v are biconnected, which contradicts our case assumption. In the following, we may assume w.l.o.g., that a, b, c, d, v are five distinct vertices.

Let P be a path from u to v in G'. Since C and C' are both connecting, P contains vertices from both components. When traversing P from u to v, let s and t denote the first and the last vertex of $C \cup C'$ that is encountered, respectively. We assume w.l.o.g. that $s \in C$ and $t \in C'$.

Let P_L be a path in C that connects s to a and let P_R be a path in C' that connects d to t. By Observation 5, P_L contains c and P_R contains b. We then obtain a $K_{2,3}$-minor of G by contracting each of $P_L[c, a]$, $P_R[d, b]$ into a single edge and by contracting the path $vuP[u, s]P_L[s, c]$ and the path $P_R[b, t]P[t, v]$ into a single edge, each. ◀

▶ **Theorem 4.** *Let C be a left (right) connecting component. It is always possible to obtain a new almost outerplanar drawing δ'_G of G from δ_G by moving only the vertices of $C \setminus \{u, v\}$ to the right (left) side.*

Proof. We assume w.l.o.g. that C is a left connecting component. Now, we determine two vertices w and w' such that a right component is a non-connecting component adjacent to C iff it lies between w and w' entirely. If u, v are not in C, by Lemma 3, C is adjacent to exactly two right connecting components C', C''(see Figure 4b). In the following, we assume that v, C', C'', u are in clockwise order. Let w be the last vertex in C' and w' be the first vertex in C'' when traversing the vertices in δ_G clockwise from v; If C contains both u and v, let w be v and w' be u; If C contains either u or v, by Lemma 3, C is adjacent to exactly one right connecting components C'. Assume w.l.o.g. that $v \in C$. Let w be the last vertex in C' when traversing the vertices in δ_G clockwise and w' be u. Observe that, due to the connectivity of G' and the outerplanarity of $\delta_{G'}$, each right component that entirely lies between w and w' is a non-connecting component adjacent to C.

Again, we want to only move the component C to the right side between w and w' without introducing any crossings. However, for simplicity of exposition, the following procedure has two phases. In the first phase, we move all the right non-connecting components connected to C to the left side “temporarily” as the procedure described in the proof of Lemma 2 such that these components are merged in C on the left side; see Figure 4c. In the second phase,

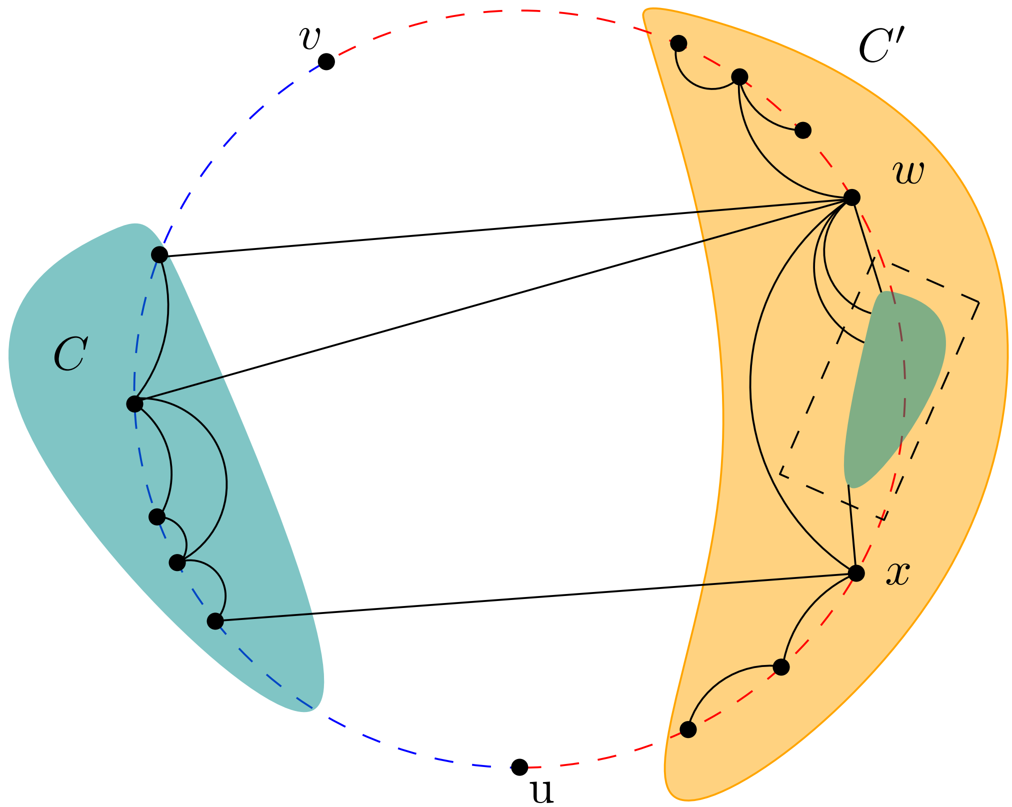


Figure 4 Case 2.2: u, v are connected but not 2-connected. **(a)** Moving a left non-connecting component C to the right side of edge uv. **(b)** A left connecting component C that is adjacent to vertices on the right side. **(c)** Moving all the right non-connecting components connected to C to the left side "temporarily" in the 1.Phase. **(d)** Moving the component C (alongside the vertices that are moved in the 1.Phase) to the right side, reversing their clockwise ordering.

we move the component $C \setminus \{u, v\}$ (alongside the vertices that are moved in the first phase) to the right side between w and w', reversing their clockwise ordering; see Figure 4d. For each right component C' that is adjacent to C, by Lemma 3, there is exactly one vertex shared by edges $E(C.C')$. Thus, there is no crossing on the right side of uv after the second phase. Furthermore, the vertices moved to the left at the first phase are in the same order as in δ_G after two reversals and they still lie between w and w'. Therefore, we could reach the same order after this two-phase procedure by only moving the vertices in C to the right side accordingly. ◀