Designing and Annotating Metro Maps with Loop Lines

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***Abstract*—Schematic metro maps provide an effective means of simplifying the geographical configuration of public rapid transportation systems. Nonetheless, travelers still find it difficult to identify routes of a specific topology on the maps because it is usually hidden behind the conventional octilinear layout of the entire map. In this paper, we present an approach to designing schematic maps with loop lines, which are drawn as circles together with annotation labels for guiding different traveling purposes. Our idea here is to formulate the aesthetic criteria as mathematical constraints in the mixed-integer programming model, which allows us to either align stations on the loop line at a grid if they are interchange stations or non-interchange stations on a circle otherwise. We then distribute the annotation labels associated with stations on the loop line evenly to the four side boundary of the map domain in order to make full use of the annotation space, while maximally avoiding intersections between leader lines and the metro network by employing a flow network algorithm. Finally, we present several experimental results generated by our prototype system to demonstrate the feasibility of the proposed approach.**

***Keywords*-Loop line, metro map, annotation label, mixed-integer programming**

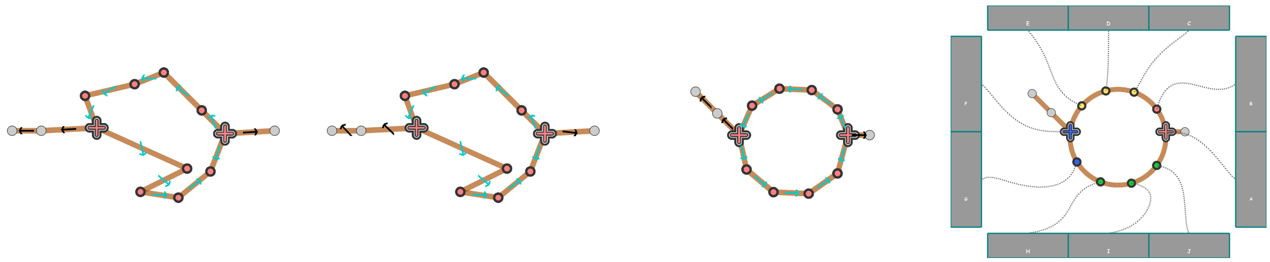
I. Introduction

A railway loop line is always significant in a transportation network due to its non-stop service along the line and high accessibility to different metro lines. Finding such a specified loop in a complex metro map is often troublesome, because this route is represented by the octilinear line segments, which reduce the trackability of the circular one [1]. Fortunately, customized maps provide us with sufficient information using a circle to represent this special route, to visually direct travelers' attention even in a global context of view. For this purpose, map deformation is commonly employed to emphasize the loop line by transforming it into a circle, while those design pamphlets are prepared by the design companies [2]. Moreover, designers also introduce large annotation labels, which contain supplementary information to this type of maps in order to guide passengers for their different traveling purposes.

On the other hand, since Beck devised the design of schematic representations for metro networks in 1933 [3], his design rules have been commonly incorporated for drawing metro networks. One significant characteristic of his design is to align metro lines with octilinear directions, including horizontal, vertical, and diagonal directions, while still preserving their geographical relative positions. Visual clarity provided by these layout schematization rules gives us perceptually plausible criteria in the sense that human is more visually sensitive to simple geometric shapes. Nonetheless, incorporating circular shapes into the octilinear layout is still technically challenging since the shape of the overall network is constrained by two different types of design rules, and thus the problem becomes further complicated if large annotation labels are also incorporated.

In this paper, we present an approach for embedding a loop line into an annotated schematic metro map. The idea here is to transform the geographical layout of a user-defined loop line into a circle while still keeping the remaining parts of the network in an octilinear layout. This can be achieved by uniformly distributing the non-interchange stations on the loop line and adaptively adjusting the orientations of the route edge segments before applying the conventional mixed-integer programming (MIP) technique. With this design for emphasizing circular routes, we can systematically annotate the stations on the map with large *external* labels such as photographs and texts around the boundary of the map domain. We then employ the *minimum-cost maximum-flow* algorithm to connect stations and their corresponding labels with leaders while suppressing possible intersections between the leader lines and map content. Our contribution lies in the formulation for customizing the shapes of the loop lines on schematic annotated metro maps.

The remainder of this paper is organized as follows: Section II provides relevant techniques of metro map layouts and their annotation. We describe the overview of the proposed system framework in Section III and then details of our approach including the formulation for designing circular routes in schematic metro maps in Section IV. An algorithm for annotating stations on loop lines is described in Section V. Section VI presents several design examples to demonstrate the feasibility of our approach, which is followed by the conclusion and future work in Section VII.



(a) (b) (c) (d)

Figure 1. Overview of the proposed approach. (a) Initial layout with a selected loop line. (b) Adjusted edge orientations after a preprocessing process. (c) Circular layout after optimization. (d) Final result with annotation labels.

II. RELATED WORK

As described previously, the line orientations in a typical schematic metro map are always aligned with octilinear directions. Early studies on schematic metro map has been done by employing conventional spring model for the networks as presented in the survey by Nöllenburg [4]. A hill climbing method is developed for generating such octilinear layouts by optimizing edge octilinearity [5]. Nonetheless, this type of energy-based method cannot always generate fully octilinear layouts on the complicated networks. Nöllenburg and Wolff introduced a mixed-integer programming (MIP) formulation, which allows us to faithfully align edges with octilinear dicections by introducing a set of linear constraints [6]. allow us to align edges to octilinear direction by introducing a set of linear equations as constraints. Wang and Chi [7] developed an accelerated optimization approach for the same objective, while the layout is smoothed before aligning edges into octilinear fashion and thus the original shape of the layout is not fully preserved. Interesting usability studies were also conducted to identify the benefits of such conventional metro layouts by the comparison with curved schematic layouts [1].

Map annotation has been one of the popular studies especially for computer scientists and cartographers [8]. Nonetheless, due to the limitation of the map domain, large labels such as thumbnail photographs are usually placed outside the boundary of the map content using *external labeling*. A typical technique, called *boundary labeling*, was proposed for placing external labels [9], which are aligned to the boundary of the map content domain. Other sophisticated methods, including those handling many-to-one label correspondence are also formulated from a mathematical point of view [10]. Moreover, the design of leader shapes was also considered in [11], [12].

Compared to general map annotation problem, only a few techniques were conducted for annotating stations on metro maps [4]. Garrido et al. [13] presented an approach for annotating stations with rectangular labels horizontally or diagonally along the lines. Wu et al. proposed several annotation techniques accompanied with different types of metro maps, including a zone-based annotation scheme [14], two-side boundary labeling [12], and spatially-efficient design [15]. Claudio and Yoon [16] introduced a dynamic labeling approach to annotating stations in a user-defined area of interest. Our technique is most relevant to the one presented by Wu et al. [12], while we improved the algorithm by relaxing the positions of stations on circular loop together with placing labels on four boundaries of the map domain.

III. OVERVIEW OF THE APPROACH

In this section, we focus on the aesthetic criteria incorporated in our approach, which is followed by a description of our system design scenario. We begin with surveying common design principles for circular layouts and aesthetic label placement along the route. According to our observation of the published hand-drawn maps, the aesthetic criteria for designing metro maps with loop lines can be summarized as follows:

(C1) **Circular route design:** Transform user-specified loop line to be a circle.

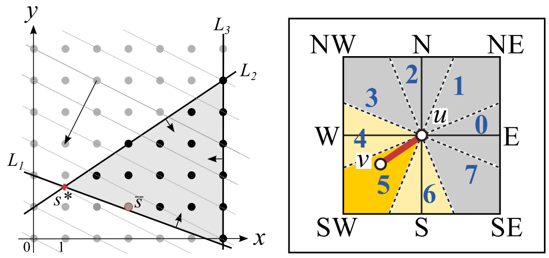
(C2) **Total leader length:** Minimize the total leader length.

(C3) **Leader intersection minimization:** Minimize intersections between the metro network and leaders.

To achieve the prescribed criteria, we summarize our system scenario for designing annotated metro maps as follows: As shown in Figure 1(a), our design begins with specifying a loop line in the metro network, and thus we uniformly distribute non-interchange stations on the route and propagate the edge orientation adjustment from the route to the remainder of the network (see arrows in Figure 1(b)). Figure 1(c) shows our result of the designed metro layout with a circle route on it. We finally apply the *four-sided boundary labeling* technique, so that the stations on the route are systematically annotated with large external labels as shown in Figure 1(d). The shapes of leader lines here are rearranged to avoid excessive intersections with metro lines in the content domain.

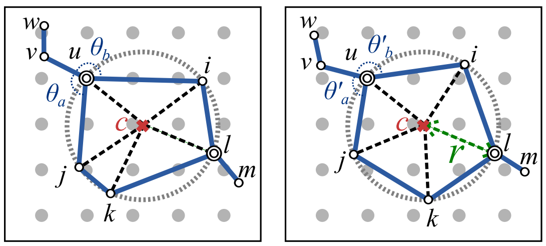
IV. METRO LAYOUT DESIGN

This section describes how we transform the loop line into a circle on the map while preserving the rest of the network in a octilinear layout. We begin with an introduction of



(a) MIP model example (b) Octilinear edge

Figure 2. MIP model on integer coordinate.



(a) Initial layout (b) Improved layout

Figure 3. Edge orientation in the metro network.

the mixed-integer programming technique, followed by the detailed formulation of our metro map design.

A. MIXED-INTEGER PROGRAMMING

Mixed-integer programming (MIP) is a well-known approach for finding the optimal solution of a linear objective function, which consists of some integer and real variables, under the linear equality and inequality constraints on these variables. Figure 2(a) shows an example of MIP model, where the solution is bounded by the linear constraints L_1, L_2, and L_3, while a set of lines correspond to level sets of the predefined objective function. We cannot easily find the optimal integer solutions $\overline{s}$ directly from optimal real-valued solutions s^*, because there is no guarantee that the two solutions are close to each other in the search space. The same problem arises in the metro map layout problem, because in order to align all the metro lines in horizontal, vertical, and 45 degree diagonal axes, we need to align every metro edge to one of the eight octilinear directions and place every station at a grid. The conventional MIP formulation allows us to effectively solve the aforementioned problem [6]. In their formulation, each edge segment is classified into one of the eight fan-shaped sectors to seek its octilinear direction in the map. For example, the orientation of the edge $\overline{uv}$ shown in Figure 2(b) can be encoded as 5 in the system, while its corresponding preceding and succeeding sectors can also be employed so as to avoid unwanted overlap among metro edges. Note that more details of conventional formulation can be founded in [6].

B. CIRCULAR ROUTE DESIGN (C1)

In our implementation, we prepare additional an angle variable for every edge to represent a rotation angle from its original orientation and minimize it in the conventional MIP formulation [6]. This provides us with enough degrees of freedom to explore an aesthetic layout with loop lines by adjusting such angle variables associated with the route. In a preprocessing stage, we first selected and reorder the stations on the loop route in a counterclockwise order by referring to their geographic positions, and uniformly distribute the non-interchange stations on the route. For example, Figure 3(a) represents the initial layout, where l, i, and u are evenly spaced in the arc length of the circle, and u, j, k, and l are as well in Figure 3(b). The orientation angle of edges in the remainder of the metro network, $\overline{uv}$ and $\overline{vw}$ for example, are then updated before applying the MIP solver. To achieve this, we compute the barycenter of the selected stations and set the temporal radius of the circle as the average distance between the barycenter and the stations initially. Then, in order to alleviate the gap in the rotation angle in the entire metro network, we propagate changes in edge orientation from the route to the remaining part. Suppose that we adjust the orientation of edge $\overline{uv}$, for example, as shown in Figure 3. We determine the edge orientation by referring to the preceding and succeeding edges of $\overline{uv}$ to keep the ratio of the two angles spanned by these edges, i.e., $\theta_a : \theta_b = \theta'_a : \theta'_b$. Once edge orientations are adjusted, we then can apply our reformulated MIP model to the network. In the MIP model, additional real variables $(x(c), y(c))$ and r are prepared to locate the center of the circle as well as the radius in the optimal layout. Here, we limit the movement of circle center within a grid unit from $(0, 0)$, while still providing enough freedom for generating radius r. Eq. (1) represents the newly defined objective function that minimizes the sum of distances between positions of the stations and their corresponding ideal positions on the circle, as follows:

$$w_{(S4)}\text{cost}_{(S4)} = w_{(S4)} \sum_{u \in V'} dist_x(u) + dist_y(u) \quad (1)$$

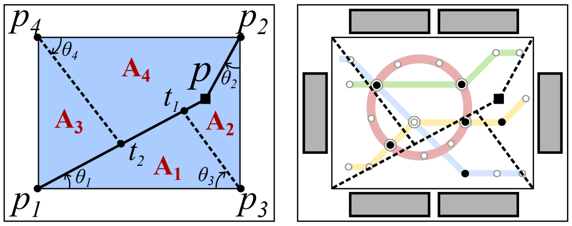
To further constrain the two variables $dist_x(u)$ and $dist_y(u)$, we retrieve the constant value θ for the station node u as the rotation angle between the vector from the temporary center to u and x-axis. Linear constraints in Eq. (2) are then introduced for this purpose,

$$\begin{cases} |x(u) - x(c) - r\cos\theta| \leq dist_x(u) \\ |y(u) - y(c) - r\sin\theta| \leq dist_y(u), \end{cases} \quad (2)$$

where $x(u)$ and $y(u)$ indicate the real values for the stations on the route, and $dist_x(u)$ and $dist_y(u)$ represents the real variable of the displacement in x and y direction, respectively. In our formulation, to generate a circle shape for the route, we relax the non-interchange stations on the route as real values instead.

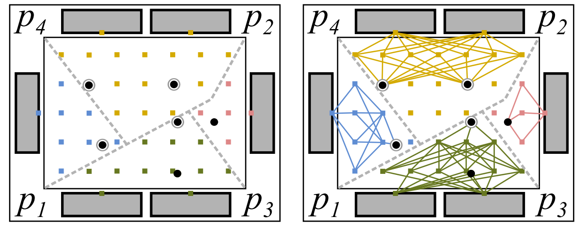
V. LABEL PLACEMENT

So far we have described how to find an initial layout of the metro network with loop lines. In this section, we explain our



(a) Domain partition (b) Label placement

Figure 4. Map domain partition for the label placement.



(a) Intermediate points (b) Graph representation

Figure 5. Network construction for guiding leader shapes.

approach for placing external labels to annotate important stations with thumbnail photographs on the lines. To achieve this, we improved the flow-network-based algorithm for placing external labels presented by Wu et al. [12].

A. *BOUNDARY LABELING*

To systematically align large external labels in the marginal space around the central map content area, we employ the *boundary labeling* technique [9] in our approach. With this technique, we thus can determine the positions of external labels without wasting too much space in the labeling area. Our algorithm also aims at designing the leader shapes by minimizing the number of intersections between every pair of leader lines together with the total leader length. In our case, we also consider minimizing the intersections between map contents and leader lines, so that users can easily discriminate the major map content from the minor leader lines. Aesthetic criteria (C2&C3) prescribed in Section III improve the readability of the metro map itself and effectively reduce visual clutter arising from line intersections (cf. Section VI). Conventional techniques for boundary labeling employ a sequence of horizontally and vertically aligned line segments as the label leader, while we again introduce curved leader styles to avoid the visual conflict with the metro network on the map.

B. *FOUR-SIDE BOUNDARY LABELING*

Compared to the two-sided boundary labeling developed by Wu et al. [12], it is more effective and consistent to place labels to the four boundary in our case, so that we can fully avoid the case where the station at the bottom is connected with the label in the top marginal area. An efficient way of achieving this is to distribute all target stations into four boundaries of the map domain, which has been introduced by Bekos et al. [9]. Figure 4(a) shows the main idea of this algorithm, where the blue area represents the content area and p_1, p_2, p_3, and p_4 represent left-bottom, right-top, right-bottom, and left-top corner points of the map domain, respectively. The algorithm begins with partitioning layout domain into two regions $A_{12}(A_1 \cup A_2)$ and $A_{34}(A_3 \cup A_4)$, while A_{12} contains $\lfloor(n+1)/2\rfloor$ labels and A_{34} contains $\lfloor n/2 \rfloor$ labels, and then partitions A_{12} into A_1 and A_2, and A_{34} into A_3 and A_4. The idea here is to alternatively turn a half-plane in counterclockwise direction from p_1, and turn another half-plane in clockwise direction from p_2 until the number of sites in region A_{12} is exactly equal to a half of the total amount. As shown in Figure 4(a), point p indicates the intersection point of the two half-planes, where $p_1 - p - p_2 - p_3(A_{12})$ forms a polygon.Once we fix the intersection point p, we then can partition polygon area $p_1 - p - p_2 - p_3(A_{12})$ and $p_1 - p - p_2 - p_4(A_{34})$ into subareas A_1, A_2, A_3, and A_4, respectively. For example, we split area A_{12} into A_1 and A_2 until A_1 contains $\lfloor(\lfloor(n+1)/2\rfloor+1)/2\rfloor$ labels and A_2 contains $\lfloor\lfloor(n+1)/2\rfloor/2\rfloor$ labels by rotating a half-plane from point p_3 in a clockwise direction. The same process can also be applied to region A_{34}. Figure 4(b) shows an example of this partitioning of the map content area.

C. *LEADER LINE DESIGN (C2&C3)*

After having partitioned the map domain into four regions, we can then connect the stations with labels through the designed curved leader lines. To find the optimal flow paths that guide the shape of leader lines, we extend the network representation for minimum-cost maximum-flow algorithm as proposed by Wu et al. [12]. In our implementation, three networks are constructed for this purpose. A residual network R is constructed for recording the weight of each edge that represents an available flow capacity, and the flow network F is used to stored the actual amount of the current flow. The edge weights of F are initialized to be 0.0, while those of R will be adjusted to seek the optimal layout of label leaders in the algorithm. To find this optimal solution, we incorporate the successive shortest path algorithm [17], which allows us to successively search an optimal flow path over the network and update the flow network F and residual network R accordingly until no valid path can be found in the flow network. In practice, the algorithm begins with finding a maximum flow augmenting path from the source node to the sink node in the residual network R, and then iteratively negates the weights and reverses the orientations of the edges on the path in R, so that the next flow stream will not be allowed to pass through these edges. Since we only offer the same amount of flow streams as the number of labels n, so each flow stream corresponds to a single leader line in this formulation.

Figure 5 shows how we construct such a graph network for laying out the leader as an example. Here, we first take relaying sample points within each of the four partitioned regions of the map content area. For example, green points in A_1 are preserved for the stations split into this area (see Figure 5(a)). Compared to the conventional approach [12], we connect each station with relaying points in its neighborhood if and only if they are in the same region as shown in Figure 5(b). We also connect all sites to a single *source* and all port nodes to a single *sink* as the conventional approach did. We also assign a weight to each edge of the network. In practice, we assign 1.0 as the weight to edges emanating from the source and sink nodes as a special case while some large weight to the rest of the edges. This allows us to effectively limit the maximum number of the flow streams in our approach.

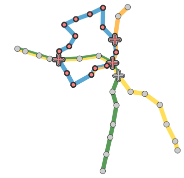
In the residual network R, we have to assign appropriate weights to edges between intermediate nodes, so that they can satisfy the following two aesthetic criteria. The first criterion is that the leader shape should be as straight as possible, and the second one is that the number of intersections between the leaders and metro lines should be reduced as many as possible. This is accomplished by assigning a different weight to edges in each of the four partitioned regions. For regions A_1 and A_4, we apply the same weight assignment in the conventional approach [12], while the edges constructed for the horizontal leaders in A_2 and A_4 are similarly defined as

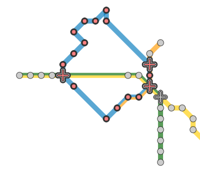
$$p\sqrt{\frac{|\vec{e}\cdot\vec{y}|}{d}} + q\sum_{j}^{n} c_j(e)\cdot\left(\frac{\vec{e}\cdot\vec{l_j}}{|\vec{e}||\vec{l_j}|}+1\right)+1, \qquad (3)$$

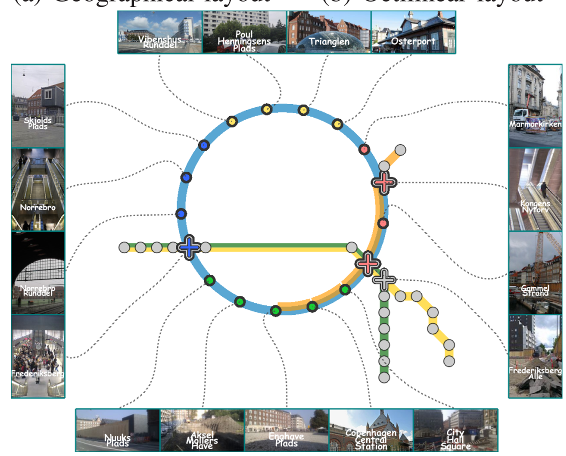
when $|\vec{e}| < d$. d is the limit vertical displacement and $l_j (j = 1, \ldots, n)$ represents the j-th line segment of the metro network. If there exists an intersection, then we set $c_j(e)$ as 1.0, and 0.0 otherwise. The first weight function penalizes the length of $\vec{e}$ projected on the horizontal axis $\vec{x}$, so that the horizontal displacement is minimized. To minimize the intersections, the second term detects those intersections between the edge $\vec{e}$ and metro line segment $\vec{l_j}$, and penalizes them, while the assigned weight is smaller if the intersection becomes more perpendicular in our case. Variables p and q here indicate the user-defined parameters for determining the relative importance between the two weight functions. Finally, a B-spline interpolation is employed to make the leader lines sufficiently smooth by referring to the intermediate points as the guide.

VI. RESULTS AND DISCUSSION

The present algorithm was implemented on a desktop PC with two Quad-Core Intel Xeon CPUs (2.4GHz, 12MB cache) and 8GB RAM. The source code was written in C++ using OpenGL for graphics rendering, OpenCV for handling images, and IBM ILOG CPLEX for solving MIP problems. The time complexity for the layout optimization in this paper is about few seconds in average during our experiments, while the one for the annotation approach is bounded by the complexity of the flow algorithm.

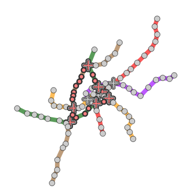

(a) Geographical layout


(b) Octilinear layout



(c) Our circular layout with annotation labels

Figure 6. Copenhagen Metro Map.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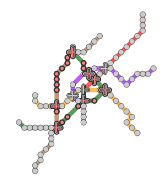
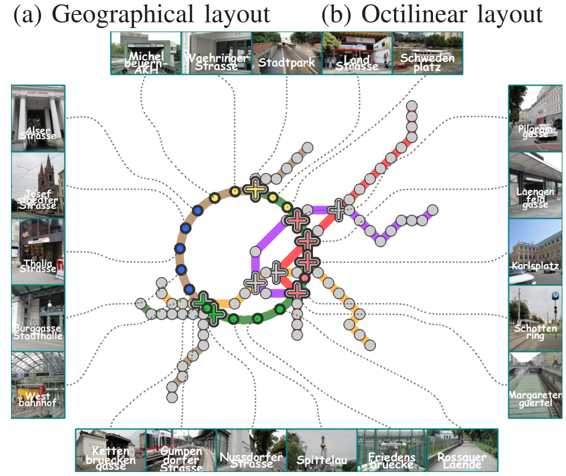

(b) Octilinear layout



(c) Our circular layout with annotation labels

Figure 7. Vienna Metro Map.

Figure 6 presents the synthesized results of Copenhagen metro map, including its geographical layout and a conventional octilinear layout obtained by the conventional MIP approach, and a optimalized layout where one circular route is emphasized. Compared to the conventional octilinear layout, as you can see, compared to the conventional octilinear layout, the circular line in our map clearly pops up from the map domain and attracts users' attention. Meanwhile, the stations on this line are systematically annotated with large external labels, which are aligned either horizontally or vertically with the boundary of the map domain and are connected to the labels via curved leaders. There are other stations on the circular line that are rendered in a difference color. For comparison, we marked stations on the loop line in red, while other colors in Figure 6(c) indicate the corresponding sides where we place the annotation labels. Figure 7 shows design examples of the Vienna metro map, where the brown-green circular route also pops up due to its schematized shape, and the labels are placed aesthetically. As for limitations of our approach, interchange stations on the circle cannot be exactly placed on the circle since other different lines go through such interchange stations and thus we have to align them on a gird to maintain the original octilinear layout, while we can distribute these errors by limiting the value $dist$ in Eq. (2). Moreover, if the edge is rotated drastically from its original orientation, the layout problem may become unsolvable due to the limitation on the topology of the input network.

VII. CONCLUSION

This paper has presented an approach to designing annotated metro maps with specific loop lines. The proposed formulation allows us to transform this specific route as a circle in the conventional MIP model, and systematically annotate the stations with the boundary labeling technique. An improved curved leader representation is achieved by partitioning the map domain into four regions before applying the minimum-cost maximum-flow algorithm to the network for guiding leader shapes. Our future extensions include formal usability study on our map design by recruiting naive users together with more sophisticated user interfaces. Additional shape patterns such as runway shapes and rounded rectangles may be considered for further customization.

ACKNOWLEDGEMENTS

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